Math 21C
Kouba
Exam 1

Please PRINT your name here : KEY

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1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. COPYING ANSWERS FROM ANOTHER STUDENT'S EXAM IS A VIOLATION OF THE UNIVERSITY HONOR CODE. HAVING SOMEONE ELSE TAKE AN EXAM FOR YOU IS A VIOLATION OF THE UNIVERSITY HONOR CODE. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. No notes, books, or classmates may be used as resources for this exam.

5. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

6. You have until 9:00 a.m. sharp to finish the exam. PLEASE STOP WRITING IMMEDIATELY when time is called and close your exam. Failure to do so may lead to points being deducted from your exam score.

7. Make sure that you have 7 pages including the cover page.

8. The following may be used on this exam.

$$(*) \int_1^{n+1} f(x)\,dx < f(1) + f(2) + \cdots + f(n) < f(1) + \int_1^n f(x)\,dx$$

$$(*)(*) \int_{n+1}^{\infty} f(x)\,dx < f(n+1) + f(n+2) + f(n+3) + \cdots < \int_n^{\infty} f(x)\,dx$$

1.) (8 pts. each) Determine whether each of the following series converges or diverges. Write clear and complete solutions including the name of the series test that you use and what your final answer is.

a.) $\sum_{n=0}^{\infty} \frac{n^2+3}{2n^2+4}$; $\lim_{n\to\infty} \frac{n^2+3}{2n^2+4} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$

$= \lim_{n\to\infty} \frac{1+\frac{3}{n^2}}{2+\frac{4}{n^2}} = \frac{1+0}{2+0} = \frac{1}{2} \neq 0$, so series diverges by Nth-term test

b.) $\sum_{n=2}^{\infty} \left(\frac{4}{n} - \frac{4}{n+1}\right)$; $S_1 = 2 - \frac{4}{3}$,

$S_2 = (2 - \cancel{\tfrac{4}{3}}) + (\cancel{\tfrac{4}{3}} - \frac{4}{4}) = 2 - \frac{4}{4}$,

$S_3 = (2 - \cancel{\tfrac{4}{3}}) + (\cancel{\tfrac{4}{3}} - \cancel{\tfrac{4}{4}}) + (\cancel{\tfrac{4}{4}} - \frac{4}{5}) = 2 - \frac{4}{5}$

$\vdots$

$S_n = 2 - \frac{4}{n+2}$, $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(2 - \frac{4}{n+2}\right)$

$= 2 - 0 = 2$, so series converges by sequence of partial sums test

c.) $\sum_{n=1}^{\infty} 2\left(\frac{5}{4}\right)^n$; series diverges by geometric series test since $r = \frac{5}{4}$, $r \geq 1$.

d.) $\sum_{n=4}^{\infty} \frac{10}{n \ln n}$; Let $f(x) = \frac{10}{x \ln x}$ which is $+$, $\downarrow$, and continuous for $x \geq 4$; $\int_4^{\infty} \frac{10}{x \ln x}\,dx = \lim_{A \to \infty} \int_4^A \frac{10}{x \ln x}\,dx$

$= \lim_{A \to \infty} 10 \ln|\ln x| \Big|_4^A = \lim_{A \to \infty} 10(\ln|\ln A| - \ln|\ln 4|)$

$= \infty - 10 \ln|\ln 4| = \infty$, so series diverges by integral test

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e.) $\sum_{n=1}^{\infty} \frac{3^n n^3}{((n+2)!)^2}$; $\lim_{n \to \infty} \frac{3^{n+1}(n+1)^3}{((n+3)!)^2} \cdot \frac{((n+2)!)^2}{3^n n^3}$

$= \lim_{n \to \infty} 3 \left(\frac{n+1}{n}\right)^3 \left[\frac{(n+2)!}{(n+3)!}\right]^2$

$= \lim_{n \to \infty} 3\left(1 + \frac{1}{n}\right)^3 \cdot \left[\frac{1}{n+3}\right]^2 = 3(1+0)^3 \cdot [0]^2$

$= 0 < 1$, so series converges by ratio test

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f.) $\sum_{n=3}^{\infty} (-1)^{n+1} \frac{1}{n(n+1)}$; let $a_n = \frac{1}{n(n+1)}$ which is $+$, $\downarrow$, and $\lim_{n \to \infty} \frac{1}{n(n+1)} = 0$, so

series converges by alternating series test g.) $\sum_{n=3}^{\infty} \frac{3+\sin n}{n+2}$, $-1 \le \sin n \le +1$, so

$2 \le 3 + \sin n \le 4$, then $\frac{2}{n+2} \le \frac{3+\sin n}{n+2}$

and $\frac{2}{n+2n} \le \frac{2}{n+2} \le \frac{3+\sin n}{n+2}$;

$\sum_{n=3}^{\infty} \frac{2}{3n} = \frac{2}{3} \sum_{n=3}^{\infty} \frac{1}{n}$ diverges by

p-series ($p = 1 \le 1$), so $\sum_{n=3}^{\infty} \frac{3+\sin n}{n+2}$

diverges by comparison test

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h.) $\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$

$S_1 = \sqrt{2} - \sqrt{1} = \sqrt{2} - 1$

$S_2 = (\cancel{\sqrt{2}} - \sqrt{1}) + (\sqrt{3} - \cancel{\sqrt{2}}) = \sqrt{3} - \sqrt{1} = \sqrt{3} - 1$

$S_3 = (\cancel{\sqrt{2}} - \sqrt{1}) + (\cancel{\sqrt{3}} - \cancel{\sqrt{2}}) + (\sqrt{4} - \cancel{\sqrt{3}})$

$= \sqrt{4} - \sqrt{1} = \sqrt{4} - 1$

$\vdots$

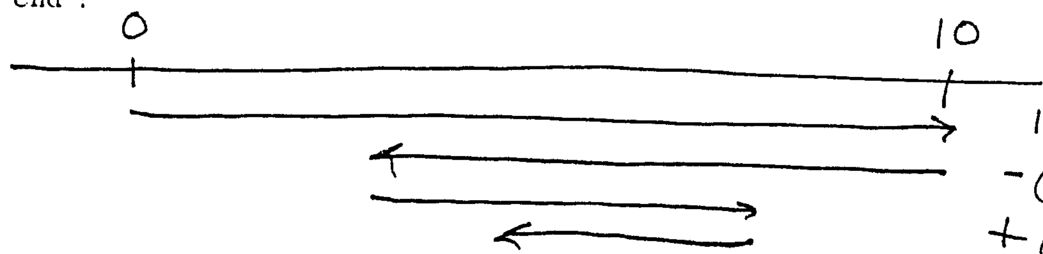
$S_n = \sqrt{n+1} - 1$, then

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (\sqrt{n+1} - 1) = \infty - 1 = \infty$

so series diverges by

sequence of partial sums test

2.) (9 pts.) A hapless snail crawls back and forth on the x-axis. The snail starts at the origin and then crawls to number 10. Then it turns around and crawls back 60% of the previous distance. Then it turns around and crawls back 60% of the previous distance. Then it turns around and crawls back 60% of the previous distance. The snail continues this back-and-forth motion without stopping. Where on the x-axis will this snail's journey "end"?



$$10$$
$$-(0.6)\cdot 10$$
$$+(0.6)^2\cdot 10$$
$$-(0.6)^3\cdot 10$$

$$X = 10 - (0.6)10 + (0.6)^2 10 - (0.6)^3 10 + (0.6)^4 10 - (0.6)^5 10 + \cdots$$

$$= 10\left(1 + (-0.6) + (-0.6)^2 + (-0.6)^3 + \cdots\right)$$

$$= 10 \cdot \frac{1}{1-(-0.6)} = 10 \cdot \frac{1}{1+\frac{3}{5}} = 10\left(\frac{5}{8}\right) = 6.25$$

3.) (9 pts.) The alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3+2}$ converges. What should n be so that the partial sum $S_n = \sum_{i=1}^{n}(-1)^{i+1}\frac{1}{i^3+2}$ estimates the exact value of the series with absolute error at most 0.0001 ?

$a_n = \frac{1}{n^3+2}$ and absolute error

$$|R_n| < a_{n+1} \le 0.0001 \rightarrow \frac{1}{(n+1)^3+2} \le 0.0001$$

$$\rightarrow (n+1)^3 + 2 \ge 10{,}000 \rightarrow (n+1)^3 \ge 9998$$

$$\rightarrow n+1 \ge 9998^{1/3} \rightarrow n \ge 9998^{1/3} - 1 \approx 20.54$$

so choose $n \ge 21$.

4.) (9 pts.) The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges. What should n be so that the partial sum $S_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$ will be at least 1000 ?

use left side of (*) :

$$1000 \le \int_1^{n+1} \frac{1}{\sqrt{x}}\,dx < \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \rightarrow$$

$$1000 \le 2\sqrt{x}\Big|_1^{n+1} \rightarrow 1000 \le 2\sqrt{n+1} - 2$$

$$\rightarrow 2\sqrt{n+1} \ge 1002 \rightarrow \sqrt{n+1} \ge 501 \rightarrow$$

$$n+1 \ge 501^2 \rightarrow n \ge 501^2 - 1 = 251{,}000 .$$

5.) (9 pts.) Givena ϵ, N-proof for $\lim_{n\to\infty} \frac{n^{1/3}-1}{3-n^{1/3}} = -1$ Let $\varepsilon > 0$ be given. Find integer N so that if $n > N$, then $\left|\frac{n^{1/3}-1}{3-n^{1/3}} - (-1)\right| < \varepsilon$, i.e., $\left|\frac{n^{1/3}-1}{3-n^{1/3}} + 1\right| < \varepsilon$. Now solve for n. Then

$$\left|\frac{n^{1/3}-1}{3-n^{1/3}} + \frac{3-n^{1/3}}{3-n^{1/3}}\right| < \varepsilon$$

iff $\frac{2}{|3-n^{1/3}|} < \varepsilon$ (assume $n > 27$)

iff $\frac{2}{-(3-n^{1/3})} < \varepsilon$

iff $\frac{2}{n^{1/3}-3} < \varepsilon$

iff $\frac{2}{\varepsilon} < n^{1/3} - 3$

iff $n^{1/3} > \frac{2}{\varepsilon} + 3$

iff $n > \left(2/\varepsilon + 3\right)^3$. Choose integer $N \ge \left(\frac{2}{\varepsilon}+3\right)^3$. Thus, if $n > N$, then

$$\left|\frac{n^{1/3}-1}{3-n^{1/3}} + 1\right| < \varepsilon .$$

QED

The following EXTRA CREDIT PROBLEM is worth 8 points. This problem is OPTIONAL.

1.) Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n}\sqrt{\frac{1+\ln n}{(\ln n)^5}} = \sum_{n=2}^{\infty} \frac{1}{n}\sqrt{\frac{1+\ln n}{(\ln n)^4 \ln n}}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n}\cdot\frac{1}{(\ln n)^2}\sqrt{1+\frac{1}{\ln n}} \; ;$$

consider fcn. $f(x) = \frac{1}{x}\cdot\frac{1}{(\ln x)^2}\cdot\sqrt{1+\frac{1}{\ln x}}$, then f is $+$, $\downarrow$, and continuous for $x \geq 2$; and

$$\int_2^{\infty} f(x)\,dx = \lim_{A\to\infty} \int_2^A \frac{1}{x}\cdot\frac{1}{(\ln x)^2}\cdot\sqrt{1+\frac{1}{\ln x}}\,dx$$

$$= \lim_{A\to\infty} -\frac{2}{3}\left(1+\frac{1}{\ln x}\right)^{3/2}\Bigg|_2^A$$

$$= \lim_{A\to\infty}\left[-\frac{2}{3}\left(1+\cancel{\frac{1}{\ln A}}^{\,0}\right)^{3/2} - -\frac{2}{3}\left(1+\frac{1}{\ln 2}\right)^{3/2}\right]$$

$$= -\frac{2}{3} + \frac{2}{3}\left(1+\frac{1}{\ln 2}\right)^{3/2} < \infty, \text{ so}$$

series converges by integral test.