Math 21C (Spring 2006)
Kouba
Exam 3

Please PRINT your name here : ---------------KEY---------------

Your Exam ID Number ------------

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. COPYING ANSWERS FROM ANOTHER STUDENT'S EXAM IS A VIOLATION OF THE UNIVERSITY HONOR CODE. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. No notes, books, or classmates may be used as resources for this exam.

5. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

6. You have until 5:00 p.m. sharp to finish the exam. PLEASE STOP WRITING IMMEDIATELY when time is called and close your exam.

7. Make sure that you have 7 pages including the cover page.

1.) (9 pts.) Let $z = 3x + \ln(x^2 + y)$. Compute the partial derivatives z_x, z_y, and z_{xx} . DO NOT SIMPLIFY your answers.

$$z_X = 3 + \frac{2X}{X^2+Y} \quad , \quad z_Y = \frac{1}{X^2+Y} \quad ,$$

$$z_{XX} = \frac{(X^2+Y)(2) - 2X(2X)}{(X^2+Y)^2}$$

2.) (9 pts.) Find and classify (relative maximum value, relative minimum value, or saddle point) each critical point for the following function :

$f(x, y) = x^3 - y^3 + 3xy$

$f_X = 3X^2 + 3Y = 3(X^2+Y) = 0 \rightarrow \boxed{Y = -X^2}$;

$f_Y = -3Y^2 + 3X = 3(-Y^2+X) = 0 \rightarrow \boxed{X = Y^2}$;

substitute $\Rightarrow Y = -(X^2) = -(Y^2)^2 = -Y^4 \Rightarrow$

$Y^4 + Y = Y(Y^3+1) = 0 \Rightarrow Y = 0$ or $Y = -1$; if $Y = 0$, then $X = 0$, so $\boxed{(0,0)}$ is critical point ; if $Y = -1$, then $X = 1$, so $\boxed{(1,-1)}$ is critical point ; then

$f_{XX} = 6X$, $f_{YY} = -6Y$, $f_{XY} = 3$;

check $(0,0)$: $D = (f_{XX})(f_{YY}) - (f_{XY})^2 = (0)(0) - (3)^2 = -9 < 0$

so $(0,0)$ determines a saddle point.

check $(1,-1)$: $D = (f_{XX})(f_{YY}) - (f_{XY})^2 = (6)(6) - (3)^2 = 27 > 0$

and $f_{XX} = 6 > 0$ so $(1,-1)$ determines a relative minimum value of $f(1,-1) = -1$.

3.) (9 pts.) Evaluate the following limit : $\lim_{(x,y)\to(1,-1)} \frac{1-\sqrt{x+y+1}}{x+y}$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{(x,y)\to(1,-1)} \frac{1-\sqrt{x+y+1}}{x+y} \cdot \frac{1+\sqrt{x+y+1}}{1+\sqrt{x+y+1}}$$

$$= \lim_{(x,y)\to(1,-1)} \frac{1-(x+y+1)}{(x+y)(1+\sqrt{x+y+1})}$$

$$= \lim_{(x,y)\to(1,-1)} \frac{-\cancel{(x+y)}}{\cancel{(x+y)}(1+\sqrt{x+y+1})}$$

$$= \frac{-1}{1+\sqrt{1-1+1}} = \frac{-1}{1+1} = \frac{-1}{2}$$

4.) (9 pts.) Verify that the following limit does not exist : $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$

<u>along path</u> $y=0$:

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{(x,y)\to(0,0)} \frac{0}{x^2} = \lim_{x\to 0} 0 = 0 \;;$$

<u>along path</u> $x=y^3$:

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{(x,y)\to(0,0)} \frac{y^3y^3}{(y^3)^2+y^6}$$

$$= \lim_{y\to 0} \frac{y^6}{y^6+y^6} = \lim_{y\to 0} \frac{y^6}{2y^6}$$

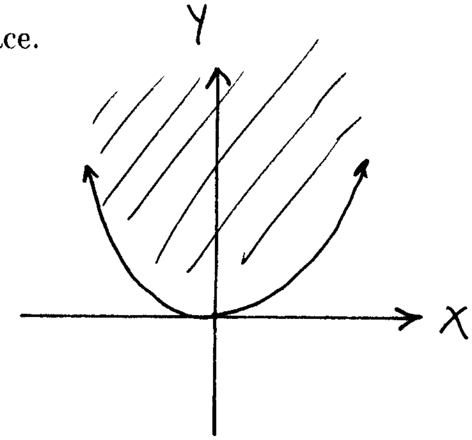
$$= \lim_{y\to 0} \frac{1}{2} = \frac{1}{2} \;;$$ so $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ DNE.

5.) Let $f(x, y) = 4 - \sqrt{y - x^2}$.

a.) (5 pts.) Determine and sketch the domain of f in 2D-space.

$Y - X^2 \geq 0 \Rightarrow Y \geq X^2$, so

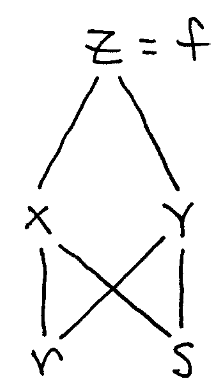
Domain is set of all points (X, Y) on or above the graph of $Y = X^2$.



b.) (5 pts.) State the range of f. BRIEFLY explain how you get your answer.

along path $X = 0$, $Z = \sqrt{Y - X^2} = \sqrt{Y}$ goes from 0 to $+\infty$. Thus, the Range $Z = 4 - \sqrt{Y - X^2}$ is $-\infty < Z \leq 4$.

6.) (9 pts.) Use the Chain Rule to find and simplify (as much as possible) the second-order partial derivative $\frac{\partial^2 z}{\partial s^2}$, where $z = f(x, y), x = 3r + s^2$, and $y = rs$.

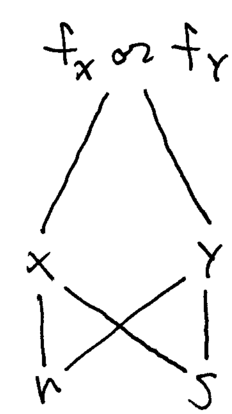


$$\frac{\partial Z}{\partial S} = f_X \cdot \frac{\partial X}{\partial S} + f_Y \cdot \frac{\partial Y}{\partial S}$$

$$= f_X \cdot (2S) + f_Y \cdot (r) \ ;$$

$$\frac{\partial^2 Z}{\partial S^2} = \frac{\partial}{\partial S}\left[\frac{\partial Z}{\partial S}\right] = \frac{\partial}{\partial S}\left[f_X \cdot (2S) + f_Y \cdot (r)\right]$$

$$= f_X \cdot \frac{\partial}{\partial S}(2S) + \frac{\partial}{\partial S}(f_X) \cdot (2S) + \frac{\partial}{\partial S}(f_Y) \cdot (r)$$



$$= f_X \cdot (2) + \left[f_{XX} \cdot \frac{\partial X}{\partial S} + f_{XY} \cdot \frac{\partial Y}{\partial S}\right](2S)$$

$$+ \left[f_{YX} \cdot \frac{\partial X}{\partial S} + f_{YY} \cdot \frac{\partial Y}{\partial S}\right] \cdot (r)$$

$$= f_X \cdot (2) + f_{XX} \cdot (2S)(2S) + f_{XY} \cdot (r)(2S)$$

$$+ f_{XY} \cdot (2S)(r) + f_{YY} \cdot (r)(r)$$

$$= 2 \cdot f_X + 4S^2 \cdot f_{XX} + 4rS \cdot f_{XY} + r^2 \cdot f_{YY}$$

7.) Consider the function given by $z = x^2 + y^2 - 4$ and its graph in 3D-space.

a.) (4 pts.) Determine all possible intercepts for this equation.

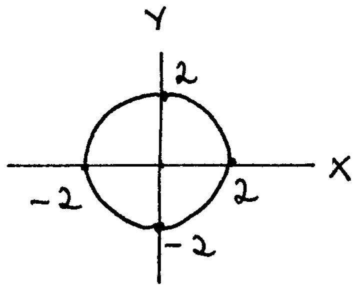
$X=0,\ Y=0 \Rightarrow Z=-4$,

$X=0,\ Z=0 \Rightarrow Y^2=4 \Rightarrow Y=\pm 2$,

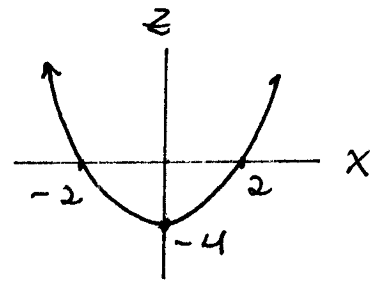
$Y=0,\ Z=0 \Rightarrow X^2=4 \Rightarrow X=\pm 2$.

b.) (4 pts.) Determine and sketch the xy-trace, xz-trace, and yz-trace.

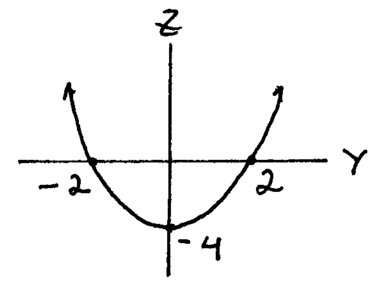
$Z=0:\ X^2+Y^2=4$



$Y=0:\ Z=X^2-4$

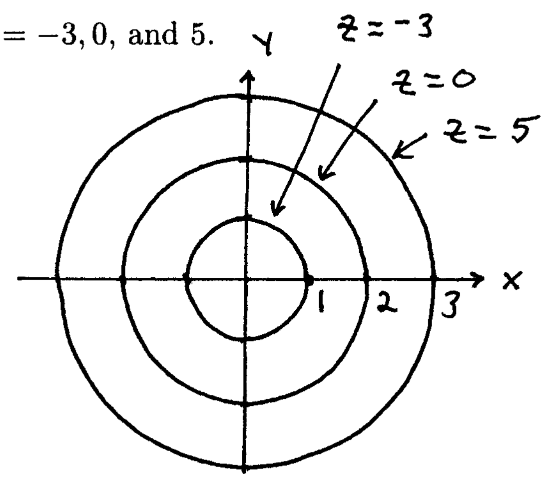


$X=0:\ Z=Y^2-4$

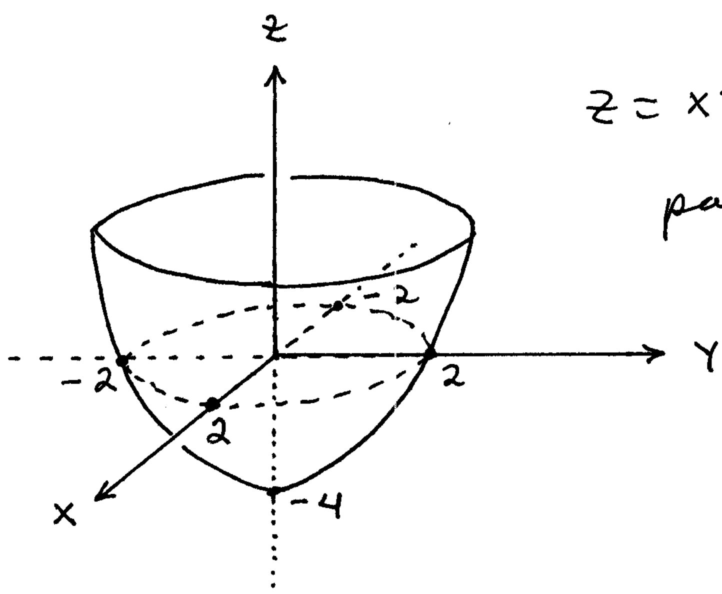


c.) (4 pts.) Sketch level curves on the same axes for $z = -3, 0$, and 5.

Z-value	level curve
-3	$X^2+Y^2=1$
0	$X^2+Y^2=4$
5	$X^2+Y^2=9$



d.) (4 pts.) Sketch the surface in 3D-space.



$Z = X^2 + Y^2 - 4$

paraboloid

8.) Consider the surface given by $z = xy^2 - 3y$ and point $P = (1, -1, 4)$ on this surface.

a.) (5 pts.) Determine a vector perpendicular to this surface at point P.

$z = xy^2 - 3y \Rightarrow \underbrace{z - xy^2 + 3y}_{f(x,y,z)} = 0$; find gradient vector $\Rightarrow$

$f_x = -y^2$, $f_y = -2xy + 3$, $f_z = 1 \Rightarrow$

$\vec{\nabla} f(1,-1,4) = f_x(1,-1,4)\vec{i} + f_y(1,-1,4)\vec{j} + f_z(1,-1,4)\vec{k} = -\vec{i} + 5\vec{j} + \vec{k}$

b.) (5 pts.) Determine an equation for the plane tangent to this surface at point P.

normal vector is $\vec{A} = -\vec{i} + 5\vec{j} + \vec{k}$ and point $P = (1,-1,4)$ so tangent plane is

$$-(x-1) + 5(y+1) + (z-4) = 0$$

c.) (5 pts.) Determine parametric equations for the line normal to this surface at point P.

Parallel vector is $\vec{A} = -\vec{i} + 5\vec{j} + \vec{k}$ and point $P = (1,-1,4)$ so line is given by

$$L: \begin{cases} x = 1 - t \\ y = -1 + 5t \\ z = 4 + t \end{cases}$$

9.) (9 pts.) Consider the function $f(x, y) = x^2 - y^2$, the point $P = (2, -1)$, and vector $\vec{A} = 3\vec{i} - 4\vec{j}$. Compute the directional derivative of f in the direction of $\vec{A}$ at point P.

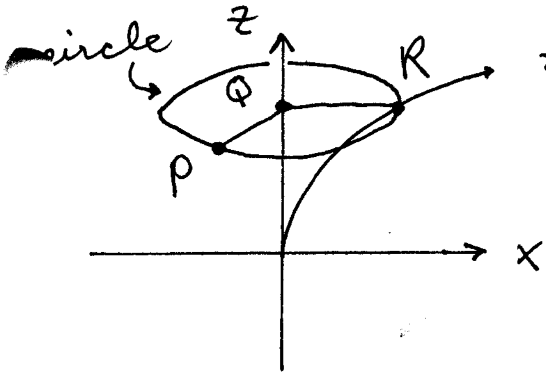
$f_x = 2x$, $f_y = -2y$ so $\vec{\nabla} f(2,-1) = f_x(2,-1)\vec{i} + f_y(2,-1)\vec{j} = 4\vec{i} + 2\vec{j}$;

for $\vec{A} = 3\vec{i} - 4\vec{j}$ the unit vector is

$\vec{u} = \frac{1}{|\vec{A}|}\vec{A} = \frac{1}{5}(3\vec{i} - 4\vec{j}) = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$; then

$$D_{\vec{u}} f(2,-1) = \vec{\nabla} f(2,-1) \cdot \vec{u}$$
$$= \overrightarrow{(4, 2)} \cdot \overrightarrow{\left(\frac{3}{5}, \frac{-4}{5}\right)} = \frac{12}{5} - \frac{8}{5} = \frac{4}{5}$$

10.) (5 pts.) Consider the graph of $z = \sqrt{x}$ in the xz-plane. Find an equation for the surface created by revolving this graph about the z-axis.



$z = \sqrt{x}$ or $x = z^2$

Points $P = (x, y, z)$, $Q = (0, 0, z)$, $R = (z^2, 0, z)$, then distances $\overline{PQ}^2 = \overline{QR}^2 \Rightarrow$

$x^2 + y^2 = (z^2)^2$

The following EXTRA CREDIT PROBLEM is worth 10 points. This problem is OPTIONAL.

1.) Find a function $z = f(x, y)$ for which

$$f_x = 3x^2y^3e^{x^3y^2} + y\sec^2(xy) + 3 \quad \text{and} \quad f_y = 2x^3y^2e^{x^3y^2} + e^{x^3y^2} + x\sec^2(xy) - 1\ ,$$

or explain why it is not possible.

$$f_x = y^3 \cdot (3x^2 e^{x^3y^2}) + y \sec^2(xy) + 3 \Rightarrow$$

$$f = y^3 \cdot \frac{1}{y^2} e^{x^3y^2} + \tan(xy) + 3x + g(y) \Rightarrow$$

(*) $$f = ye^{x^3y^2} + \tan(xy) + 3x + g(y) \Rightarrow$$

$$f_y = y \cdot 2y \cdot x^3 e^{x^3y^2} + e^{x^3y^2} + x\sec^2(x, y) + g'(y)$$

$$= 2x^3y^2e^{x^3y^2} + e^{x^3y^2} + x\sec^2(xy) + g'(y)$$

$$\Rightarrow g'(y) = -1 \Rightarrow g(y) = -y\ ,$$ so

$$f(x,y) = ye^{x^3y^2} + \tan(xy) + 3x - y$$ works.