Section 16.3
Thomas Calculus
11th Ed.

Gradient Fields (Conservative Vector Fields) and Path Independence

Recall: Consider the scalar function $w = f(x,y,z)$. Its gradient field is the vector field

$$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k} .$$

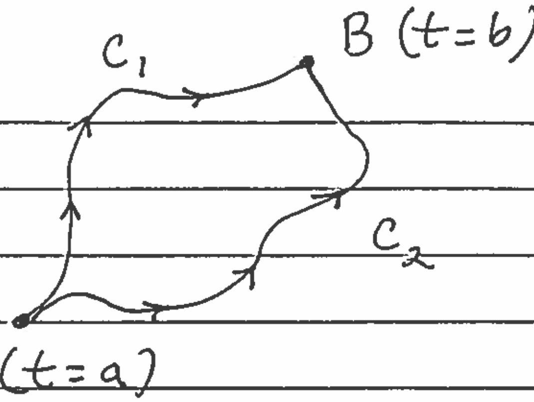
Example I: If $f(x,y,z) = xy + yz + xz$, then its gradient field is

$$\vec{\nabla} f(x,y,z) = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k} .$$

Of course not all vector fields are gradient fields. This is just one type of vector field. However, gradient fields have very special properties.

Recall: Let $\vec{F}$ be a vector field defined on path $C : \vec{r}(t)$ for $a \le t \le b$. The work done by $\vec{F}$ on path C is

$$\text{Work} = \int_C \vec{F} \cdot \vec{T} \, ds .$$



Note: In general,

$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds \neq \int_{C_2} \vec{F}\cdot\vec{T}\,ds$$

for different paths C_1 and C_2 between points A and B. However, for some vector fields $\vec{F}$

$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{T}\,ds.$$

for all paths C_1 and C_2 between points A and B. (See problem 12, p. 1142.)

Def: Let $\vec{F}$ be a vector field defined on a region D and let A and B be any two points in D. If

$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{T}\,ds$$

for any two paths C_1 and C_2 from point A to point B, then we call $\vec{F}$ a <u>conservative</u> <u>vector field</u>. We say that the work integral from point A to point

B is path independent.

Ex: It will be shown later that the vector field in Example A is conservative.

Recall: (FTC from Math 21B)
If $f'(x) = F(x)$, then
$$\int_a^b F(x)\,dx = f(x)\Big|_a^b = f(b) - f(a).$$

Recall: (Chain Rule) If $w = f(x,y,z)$ and $x = g(t)$, $y = h(t)$, and $z = k(t)$, then
$$\frac{df}{dt} = f_x \cdot \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \cdot \frac{dz}{dt}.$$

Fundamental Theorem for Line Integrals: Let $w = f(x,y,z)$ be a scalar function and let $\vec{\nabla} f = f_x \cdot \vec{i} + f_y \cdot \vec{j} + f_z \vec{k}$ be its gradient field defined on path C: $\vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j} + k(t)\,\vec{k}$ for $a \le t \le b$. Let $A = \vec{r}(a)$ and $B = \vec{r}(b)$.
Then $\displaystyle\int_C \vec{\nabla} f \cdot \vec{T}\,ds = f(\vec{r}(t))\Big|_a^b = f(B) - f(A).$

Proof: $\int_C \vec{\nabla} f(P) \cdot \vec{T}\, ds = \int_C \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$= \int_a^b (f_x \vec{i} + f_y \vec{j} + f_z \vec{k}) \cdot (g'(t)\vec{i} + h'(t)\vec{j} + k'(t)\vec{k})\, dt$

$= \int_a^b (f_x \cdot g'(t) + f_y \cdot h'(t) + f_z \cdot k'(t))\, dt$

$= \int_a^b \frac{d}{dt} f(g(t), h(t), k(t))\, dt$ (Chain Rule)

$= f(g(t), h(t), k(t)) \Big|_a^b$ (FTC)

$= f(g(b), h(b), k(b)) - f(g(a), h(a), k(a))$

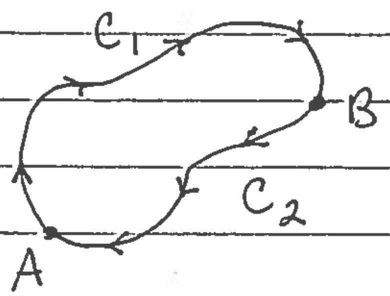
$= f(B) - f(A)$.

Fact: $\int_{t=a}^{t=b} f(P)\, ds = - \int_{t=b}^{t=a} f(P)\, ds$.

Theorem 1: A vector field $\vec{F}$ is conservative iff $\oint_C \vec{F} \cdot \vec{T}\, ds = 0$ for every closed curve C.

Proof: $(\Rightarrow)$ Assume $\vec{F}$ is conservative. Show that $\oint_C \vec{F} \cdot \vec{T}\, ds = 0$ for every closed curve C. Let C be a closed path starting at point A. Let B be another point on path C. Let C_1 be

C



the path from A to B and let C_2 be the path from B to A. Since $\vec{F}$ is conservative, we know that all work integrals from point A to point B are equal, so that

$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds = -\int_{C_2} \vec{F}\cdot\vec{T}\,ds .$$

Then

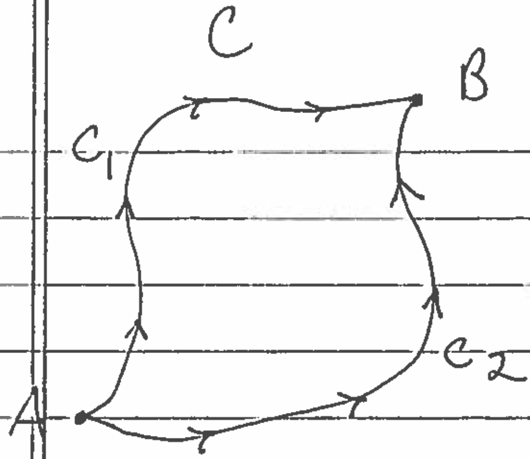
$$\oint_C \vec{F}\cdot\vec{T}\,ds = \int_{C_1} \vec{F}\cdot\vec{T}\,ds + \int_{C_2} \vec{F}\cdot\vec{T}\,ds$$

$$= -\int_{C_2} \vec{F}\cdot\vec{T}\,ds + \int_{C_2} \vec{F}\cdot\vec{T}\,ds = 0 .$$

($\Leftarrow$) Assume $\oint_C \vec{F}\cdot\vec{T}\,ds = 0$ for every closed curve C. Let A and B be any two points. Show that $\vec{F}$ is conservative, i.e., show that

$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{T}\,ds$$

for any two paths C_1 and C_2 from point A to point B. Consider the



closed path C from A to B (along C_1) and then from B to A (reverse of C_2). Then

$$0 = \oint_C \vec{F} \cdot \vec{T}\, ds = \int_{C_1} \vec{F} \cdot \vec{T}\, ds + \left(- \int_{C_2} \vec{F} \cdot \vec{T}\, ds \right)$$

$$\rightarrow \int_{C_1} \vec{F} \cdot \vec{T}\, ds = \int_{C_2} \vec{F} \cdot \vec{T}\, ds .$$

Theorem 2: Let $\vec{F}$ be a gradient field, i.e., assume that $\vec{F} = \vec{\nabla} f$ for some scalar function $w = f(x,y,z)$. Then $\vec{F}$ is conservative.

Proof: Let path C be given by $\vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j} + k(t)\,\vec{k}$ for $a \le t \le b$, where $A = \vec{r}(a)$ and $B = \vec{r}(b)$. Then

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{\nabla} f \cdot \vec{T}\, ds$$

$$= f(\vec{r}(t)) \Big|_a^b = f(B) - f(A).$$

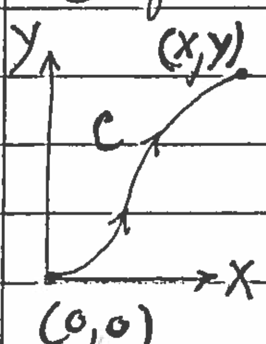
(by Fundamental Theorem for Line Integrals)

The answer implies that only the endpoints of path C matter, so that the work integral from point A to point B is path independent. This means $\vec{F}$ is conservative.

Theorem 3 : Assume that $\vec{F}$ is a conservative vector field. Then $\vec{F}$ must also be a gradient field, i.e., there is some scalar function $w = f(x,y,z)$ so that

$$\vec{F} = \vec{\nabla} f .$$

Proof: (in 2D-space) Assume that $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ is a conservative vector field. Define scalar function f as follows: for each point (x,y) select a path C from $(0,0)$ to (x,y). Define

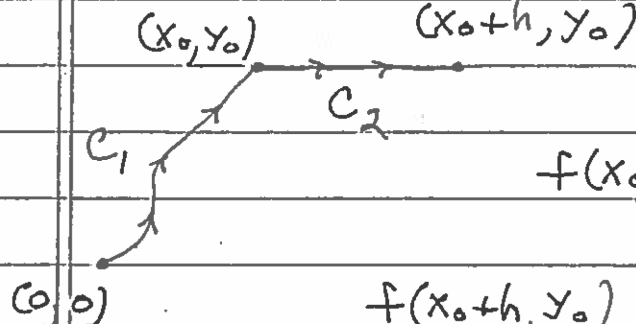


$$f(x,y) = \int_C \vec{F} \cdot \vec{T}\, ds .$$

We will show that $\vec{\nabla} f = \vec{F}$, i.e., that $f_x = M$ and $f_y = M$.

Let (x_0, y_0) be a fixed point and compute $\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$.

Let C_1 be any path from $(0,0)$ to (x_0, y_0) and let C_2 be the straight path from (x_0, y_0) to (x_0+h, x_0). Let C be paths C_1 and C_2 combined. Then



$$f(x_0, y_0) = \int_{C_1} \vec{F} \cdot \vec{T}\, ds, \text{ and}$$

$$f(x_0+h, y_0) = \int_{C} \vec{F} \cdot \vec{T}\, ds$$

$$= \int_{C_1} \vec{F} \cdot \vec{T}\, ds + \int_{C_2} \vec{F} \cdot \vec{T}\, ds$$

$$= \int_{C_1} \vec{F} \cdot \vec{T}\, ds + \int_{C_2} \vec{F} \cdot \vec{r}\,'(t)\, dt$$

$$= \int_{C_1} \vec{F} \cdot \vec{T}\, ds + \int_{C_2} (M\vec{i} + N\vec{j})\left(\frac{dx}{dt}\vec{i} + \cancelto{0}{\frac{dy}{dt}}\vec{j}\right) dt$$

$$= \int_{C_1} \vec{F} \cdot \vec{T}\, ds + \int_{C_2} M \frac{dx}{dt}\, dt$$

$$= \int_{C_1} \vec{F} \cdot \vec{T}\, ds + \int_{x_0}^{x_0+h} M(x, y_0)\, dx \text{ ; then}$$

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

$$= \lim_{h \to 0} \frac{\int_{x_0}^{x_0+h} M(x, y_0)\, dx}{h} = M(x_0, y_0) \text{ ; i.e.,}$$

SEE (*) below

$$\frac{\partial f}{\partial x}(x_0, y_0) = M(x_0, y_0) \text{ ; similarly,}$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = N(x_0, y_0) \text{. This}$$

establishes that $\vec{F} = \vec{\nabla} f$.

(*) (Recall: (FTC from Math 21B)

$$\frac{d}{dx} \int_a^x G(t)\, dt = G(x) \text{ ; and}$$

$$\frac{d}{dx} \int_a^x G(t)\, dt = \lim_{h \to 0} \frac{\int_a^{x+h} G(t)\, dt - \int_a^x G(t)\, dt}{h}$$

$$= \lim_{h \to 0} \frac{\int_x^{x+h} G(t)\, dt}{h} = G(x) \text{ .)}$$

Question : Is there a test to determine if a given vector field $\vec{F}$ is conservative ? ... YES !

Component Test for Conservative Fields : I.) Let

$$\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$$

be a vector field. Then $\vec{F}$ is CONSERVATIVE iff

$$P_y = N_z \ , \ M_z = P_x \text{, and } N_x = M_y \ .$$

II.) Let $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ be a vector field. Then $\vec{F}$ is CONSERVATIVE iff

$$N_x = M_y \quad .$$

Proof: I.) ($\Rightarrow$) If $\vec{F}$ is conservative, then there is a scalar function f satisfying $\vec{F} = \vec{\nabla} f$; thus

$$\vec{F} = M\vec{i} + N\vec{j} + P\vec{k} = f_x \cdot \vec{i} + f_y \cdot \vec{j} + f_z \cdot \vec{k} \ .$$

It follows that

$$P = f_z \rightarrow P_y = (f_z)_y = (f_y)_z = N_z \ ;$$

$$M = f_x \rightarrow M_z = (f_x)_z = (f_z)_x = P_x \ ;$$

$$N = f_y \rightarrow N_x = (f_y)_x = (f_x)_y = M_y \ .$$

($\Leftarrow$) Omitted.

II.) If $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$

then $N_z = 0$, $M_z = 0$, and $P = 0$ so that $P_x = P_y = 0$. The result follows.

<u>Definition</u>: We know that if $\vec{F}$ is a conservative vector field, then $\vec{F} = \vec{\nabla} f$ for some scalar function f. We call f a <u>potential function</u> for $\vec{F}$.

<u>Example</u>: Show that each vector field $\vec{F}$ is conservative and then find a potential function f.

1.) $\vec{F}(x,y) = (xy^2)\vec{i} + (x^2y+1)\vec{j}$

2.) $\vec{F}(x,y,z) = (y)\vec{i} + (x)\vec{j} + (2z)\vec{k}$

3.) $\vec{F}(x,y) = (e^x \sin y + \tan y)\vec{i} + (e^x \cos y + x \sec^2 y)\vec{j}$

4.) $\vec{F}(x,y,z) = \left(\dfrac{2x + 3x^2y^2}{z^2}\right)\vec{i} + \left(\dfrac{2x^3y}{z^2}\right)\vec{j} - \left(\dfrac{1 + 2x^2 + 2x^3y^2}{z^3}\right)\vec{k}$

1.) $F(x,y) = \underset{\substack{\uparrow \\ M}}{(xy^2)}\vec{i} + \underset{\substack{\uparrow \\ N}}{(x^2y+1)}\vec{j}$, then

$M_y = 2xy = N_x$, so this vector field is conservative; so

$f_x = xy^2 \xrightarrow{\int_x} f = \frac{1}{2}x^2y^2 + g(y) \xrightarrow{D_y}$

$f_y = x^2y + g'(y) = x^2y + 1 \longrightarrow$

$g'(y) = 1 \xrightarrow{\int} g(y) = y + \cancelto{0}{c}$, so

$\boxed{f(x,y) = \frac{1}{2}x^2y^2 + y}$ is a potential function.

3.) $\vec{F}(x,y) = \underbrace{(e^x \sin y + \tan y)}_{M}\,\vec{i} + \underbrace{(e^x \cos y + x \sec^2 y)}_{N}\,\vec{j}$,

then

$$M_y = e^x \cos y + \sec^2 y = N_x,$$ so this vector field is conservative; so

$$f_x = e^x \sin y + \tan y \xrightarrow{S_x}$$

$$f = e^x \sin y + x \tan y + g(y) \xrightarrow{D_y}$$

$$f_y = e^x \cos y + x \sec^2 y + g'(y)$$

$$= e^x \cos y + x \sec^2 y \rightarrow g'(y) = 0 \rightarrow$$

$g(y) = C^{\nearrow 0}$, then

$\boxed{f(x,y) = e^x \sin y + x \tan y}$ is a potential function.

2.) $\vec{F}(x,y,z) = \underbrace{(y)}_{M}\,\vec{i} + \underbrace{(x+1)}_{N}\,\vec{j} + \underbrace{(2z-3)}_{P}\,\vec{k}$,

then

$P_y = 0 = N_z$, $M_z = 0 = P_x$, and $N_x = 1 = M_y$,

so this vector field is conservative;

$$P_y = -\frac{0 + 2x^3 \cdot (2y)}{z^3} = -\frac{4x^3y}{z^3} \text{ and}$$

$$N_z = 2x^3y \cdot (-2z^{-3}) = -\frac{4x^3y}{z^3}, \text{ so}$$

$$P_y = N_z \ ;$$

$$N_x = \frac{6x^2y}{z^2} \text{ and}$$

$$M_y = \frac{0 + 3x^2 \cdot (2y)}{z^2} = \frac{6x^2y}{z^2}, \text{ so}$$

$$N_x = M_y \ ;$$

$$M_z = (2x + 3x^2y^2) \cdot (-2z^{-3})$$

$$= \frac{-4x - 6x^2y^2}{z^3} \text{ and}$$

$$P_x = -\frac{4x + 6x^2y^2}{z^3}, \text{ so}$$

$M_z = P_x$; so this vector field is conservative ;

so $f_x = y \xrightarrow{\int x} f = xy + g(y, z) \xrightarrow{D_y}$

$f_y = x + g_y(y, z) = x + 1 \rightarrow g_y(y, z) = 1 \xrightarrow{\int y}$

$g(y, z) = y + k(z) \longrightarrow f = xy + y + k(z)$

$\xrightarrow{D_z} f_z = 0 + 0 + k'(z) = 2z - 3 \xrightarrow{\int}$

$k(z) = z^2 - 3z + \cancel{C}^{\,0}$, then

$\boxed{f(x, y, z) = xy + y + z^2 - 3z}$ is a

potential function.

4.) $\vec{F}(x, y, z) = \left(\dfrac{2x + 3x^2y^2}{z^2}\right)\vec{i}$

$+ \left(\dfrac{2x^3y}{z^2}\right)\vec{j} - \left(\dfrac{1 + 2x^2 + 2x^3y^2}{z^3}\right)\vec{k}$ with

$M = \dfrac{2x + 3x^2y^2}{z^2}$, $N = \dfrac{2x^3y}{z^2}$, and

$P = -\dfrac{2x^2 + 2x^3y^2}{z^3}$; then

$$f_y = \frac{2x^3y}{z^2} \xrightarrow{\int_y} f = \frac{2x^3 \cdot \frac{1}{2} y^2}{z^2} + g(x,z)$$

$$= \frac{x^3y^2}{z^2} + g(x,z) \xrightarrow{D_x}$$

$$f_x = \frac{3x^2y^2}{z^2} + g_x(x,z)$$

$$= \frac{2x}{z^2} + \frac{3x^2y^2}{z^2} \longrightarrow g_x(x,z) = \frac{2x}{z^2}$$

$$\xrightarrow{\int_x} g(x,z) = \frac{x^2}{z^2} + k(z) \longrightarrow$$

$$f = \frac{x^3y^2}{z^2} + \frac{x^2}{z^2} + k(z) \xrightarrow{D_z}$$

$$f_z = x^3y^2 \cdot (-2z^{-3}) + x^2 \cdot (-2z^{-3}) + k'(z)$$

$$= \frac{-2x^3y^2}{z^3} + \frac{-2x^2}{z^3} + k'(z)$$

$$= \frac{-1}{z^3} + \frac{-2x^2}{z^3} + \frac{-2x^3y^2}{z^3} \longrightarrow$$

$$k'(z) = \frac{-1}{z^3} = -z^{-3} \xrightarrow{\int} k(z) = \frac{1}{2} z^{-2} + \cancel{C}^{0}$$

$$\rightarrow \quad \boxed{f(x,y,z) = \frac{x^3y^2 + x^2}{z^2} + \frac{1}{2z^2}}$$

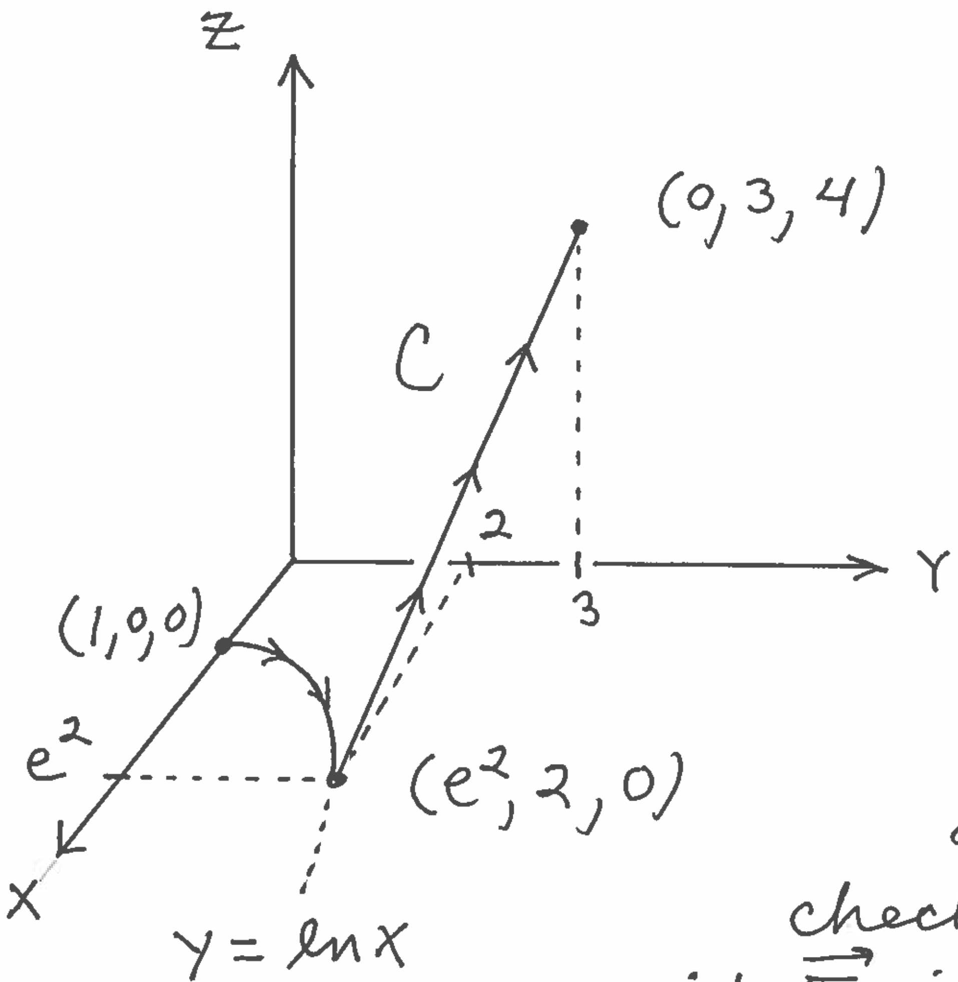
is a potential function.

<u>Example</u>: Consider the vector field

$$\vec{F}(x,y,z) = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$$

and the path C in 3D-Space described as follows:

C starts at point $(1,0,0)$, follows the graph of $y = \ln x$ in the xy-plane to point $(e^2, 2, 0)$, and then follows a straight line to the point $(0,3,4)$, where path C ends. Compute the work done by $\vec{F}$ along path C.



Here is path C:

Let's first check to see if $\vec{F}$ is a conservative vector field;

$$\vec{F}(X,Y,Z) = \underset{\underset{M}{\uparrow}}{(Y+Z)}\,\vec{i} + \underset{\underset{N}{\uparrow}}{(X+Z)}\,\vec{j} + \underset{\underset{P}{\uparrow}}{(X+Y)}\,\vec{k}$$

$P_Y = 1 = N_Z$, $N_X = 1 = M_Y$, and $M_Z = 1 = P_X$, so $\vec{F}$ is conservative. Now let's find a potential function f and apply The Fundamental

Theorem for Line Integrals.
Then

$$f_x = y + z \xrightarrow{\int_x} f = xy + xz + g(y,z)$$

$$\xrightarrow{D_y} f_y = x + 0 + g_y(y,z) = x + z \longrightarrow$$

$$g_y(y,z) = z \xrightarrow{\int_y} g(y,z) = yz + k(z)$$

$$\longrightarrow f = xy + xz + yz + k(z) \xrightarrow{D_z}$$

$$f_z = 0 + x + y + k'(z) = x + y \longrightarrow$$

$$k'(z) = 0 \xrightarrow{\int} k(z) = \cancel{c}^{\,0},$$ then

$$f(x,y,z) = xy + xz + yz;$$ apply Theorem getting

$$\text{Work} = \int_C \vec{F} \cdot \vec{T}\, ds$$

$$= f(x,y,z) \Big|_{(1,0,0)}^{(0,3,4)}$$

$$= [(0)(3) + (0)(4) + (3)(4)] - [(1)(0) + (1)(0) + (0)(0)] = 12$$