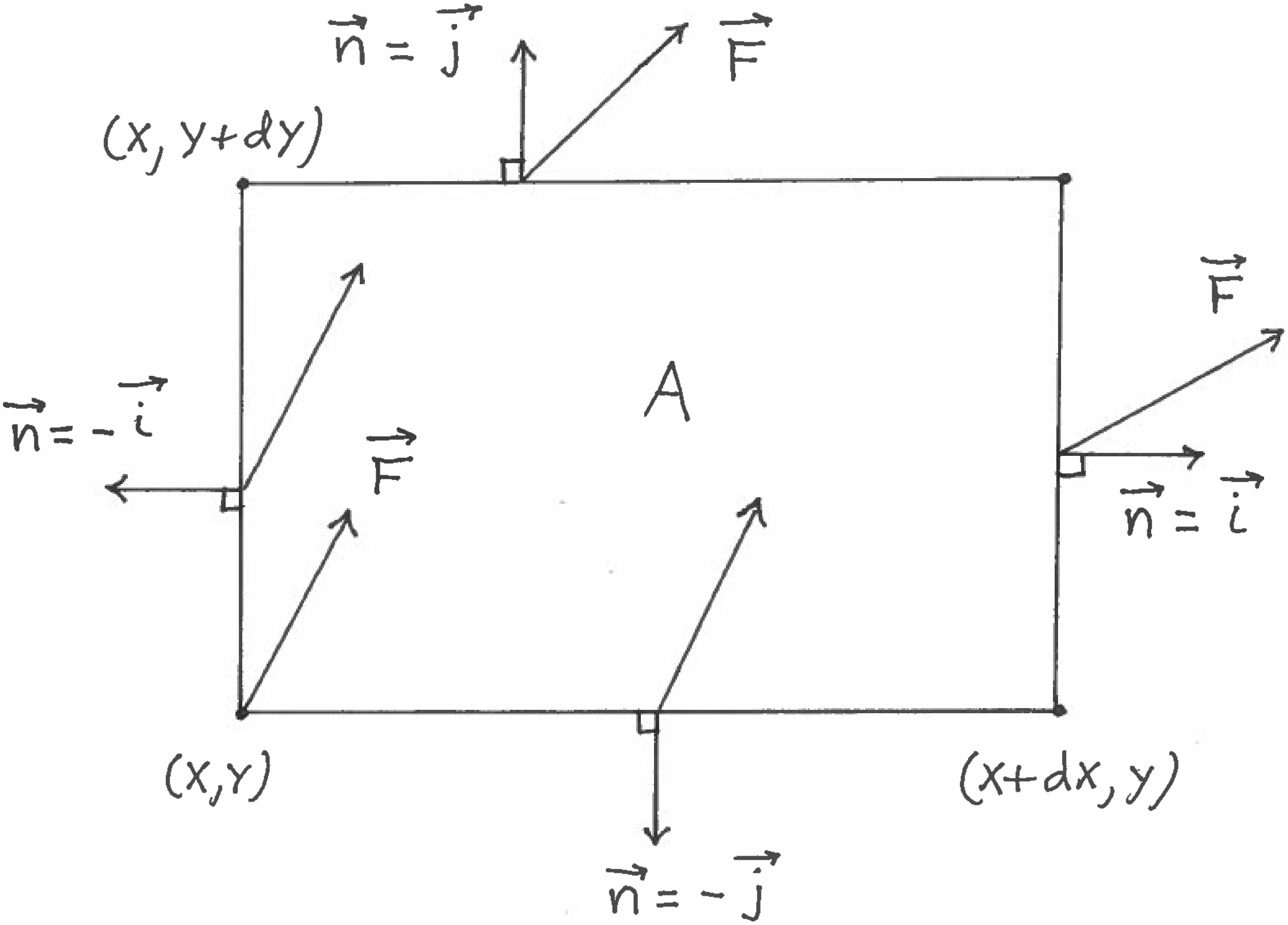
The Divergence of a Vector Field

These notes will help explain why the definition of the divergence of a vector field is a sensible one.

Assume that an incompressible fluid over a region R in the XY-plane is represented by a velocity vector field

$$\vec{F}(X,Y) = \underbrace{\underset{\substack{\uparrow \\ \text{fluid density:} \\ \text{gm./cm.}^2}}{\delta} \cdot \underset{\substack{\uparrow \\ \text{fluid velocity:} \\ \text{cm./sec.}}}{\vec{V}(t)}}_{\frac{\text{gm.}}{(\text{cm.})(\text{sec.})}}$$

Consider a small rectangle A in region R at point (X,Y) of dimensions dx by dy cm. Let's estimate the <u>flux</u> of $\vec{F}$ across rectangle A:



assume that $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$. We can now estimate the flux of $\vec{F}$ across the boundary of A by adding the flux across each of the four edges:

<u>edge</u>:

bottom: $\left(\vec{F}(x,y)\cdot(-\vec{j})\right) dx = -N(x,y)\, dx$

right: $\left(\vec{F}(x+dx,y)\cdot\vec{i}\right) dy = M(x+dx,y)\, dy$

top: $\left(\vec{F}(x,y+dy)\cdot\vec{j}\right) dx = N(x,y+dy)\, dx$

left: $\left(\vec{F}(x,y)\cdot(-\vec{i})\right) dy = -M(x,y)\, dy$,

then flux across rectangle A is $\approx$

$$M(x+dx,y)\,dy - M(x,y)\,dy$$
$$+ \left(N(x,y+dy)\,dx - N(x,y)\,dx\right)$$

$$= \left(M(x+dx,y) - M(x,y)\right) dy$$
$$+ \left(N(x,y+dy) - N(x,y)\right) dx$$

$$= \frac{M(x+dx,y) - M(x,y)}{dx} \cdot dy\, dx$$

$$+ \frac{N(x,y+dy) - N(x,y)}{dy} \cdot dx\, dy$$

$$\approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dy\, dx \; ; \text{ now}$$

divide by the area of A, getting

$$\text{divergence} = \frac{\text{flux across } A}{\text{area of } A}$$

$$= \frac{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dy\, dx}{dy\, dx}$$

$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \qquad \frac{\text{gm.}}{(\text{sec.})(\text{cm.}^2)}$$