Example: Consider the velocity vector field given by

$$\vec{F}(x,y) = (x)\vec{i} + (y)\vec{j}$$

for a fluid with density units $\frac{gm.}{cm^2}$ and velocity units $\frac{cm.}{min.}$.

Find the Flux of $\vec{F}$ across the circle $x^2+y^2=4$ (oriented counter-clockwise).

$$C: \begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases} \text{ for } 0 \le t \le 2\pi,$$

then Flux of $\vec{F}$ across C is

$$\text{Flux} = \oint_C M\,dy - N\,dx$$

$$= \int_C \left[(x)\cdot\frac{dy}{dt} - (y)\frac{dx}{dt}\right] dt$$

$$= \int_0^{2\pi} \left[(2\cos t)(2\cos t) - (2\sin t)(-2\sin t)\right] dt$$

$$= \int_0^{2\pi} [4\cos^2 t + 4\sin^2 t]\, dt$$

$$= \int_0^{2\pi} 4(\cos^2 t + \sin^2 t)\, dt$$

$$= \int_0^{2\pi} 4(1)\, dt = 4t \Big|_0^{2\pi}$$

$$= 8\pi \text{ gm./min.}$$

<u>Example</u>: Find the $\boxed{\text{Flow}}$ of $\vec{F}$ in the previous example along the same circular path C.

$$\text{Flow} = \oint_C M\,dx + N\,dy$$

$$= \oint_C \left[(X)\left(\frac{dX}{dt}\right) + (Y)\left(\frac{dY}{dt}\right)\right] dt$$

$$= \int_0^{2\pi} [(2\cos t)(-2\sin t) + (2\sin t)(2\cos t)]\, dt$$

$$= \int_0^{2\pi} [-4\sin t\cos t + 4\sin t\cos t]\, dt$$

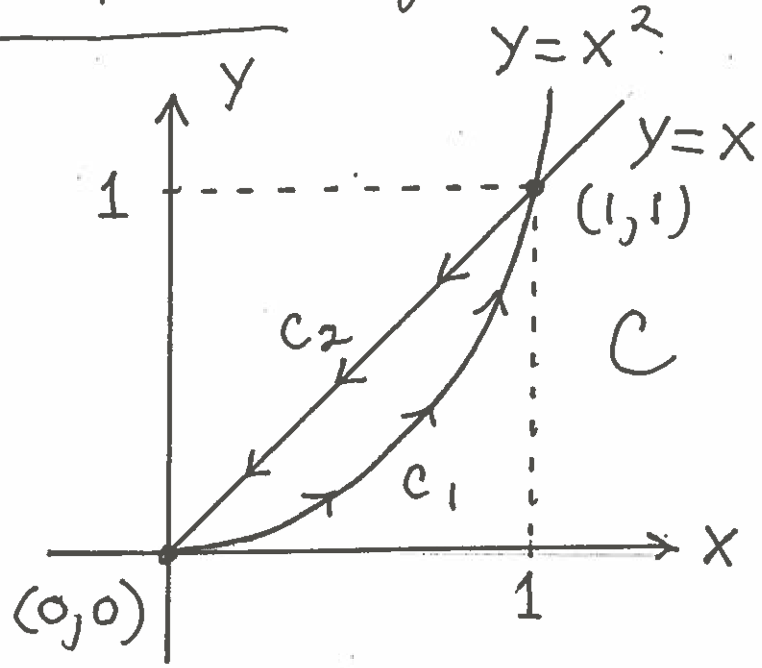
$$= \int_0^{2\pi} 0\, dt = 0 \text{ gm./min.}$$

Example: Consider the velocity vector field given by

$$\vec{F} = (x+y)\,\vec{i} - (y)\,\vec{j}$$

for a fluid with density units $\frac{kg.}{m.^2}$ and velocity units $\frac{m.}{hr.}$.

Find the $\boxed{\text{Flux}}$ of $\vec{F}$ across path C given below.



$$C_1 : \begin{cases} x = t \\ y = t^2 \end{cases} \quad \text{for } 0 \le t \le 1$$

$$C_2 : \begin{cases} x = 1-t \\ y = 1-t \end{cases} \quad \text{for } 0 \le t \le 1$$

$$\text{Flux} = \oint_C \vec{F} \cdot \vec{n}\, ds = \oint_C M\,dy - N\,dx$$

$$= \oint_{C_1} M\,dy - N\,dx + \oint_{C_2} M\,dy - N\,dx$$

$$= \oint_{C_1} \left[(x+y)\left(\frac{dy}{dt}\right) - (-y)\left(\frac{dx}{dt}\right) \right] dt$$

$$+ \oint_{C_2} \left[(x+y)\left(\frac{dy}{dt}\right) - (-y)\left(\frac{dx}{dt}\right) \right] dt$$

$$= \int_0^1 \left[(t+t^2)(2t) + (t^2)(1) \right] dt$$

$$+ \int_0^1 \left[((1-t)+(1-t))(-1) + (1-t)(-1) \right] dt$$

$$= \int_0^1 \left[3t^2 + 2t^3 \right] dt$$

$$+ \int_0^1 \left[-2 + 2t + t - 1 \right] dt$$

$$= \left[\quad t^3 + \frac{1}{2} t^4 \right] \Big|_0^1 + \int_0^1 (3t-3)\, dt$$

$$= \left(1 + \frac{1}{2} \right) + \left(\frac{3}{2} t^2 - 3t \right) \Big|_0^1$$

$$= \frac{3}{2} + \left(\frac{3}{2} - 3 \right)$$

$$= 3 - 3 = \quad 0 \quad \text{kg./hr.}$$