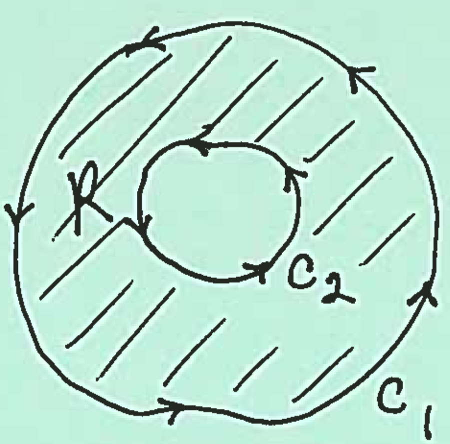
Section 16.4
Thomas Calculus
11th Ed.

Green's Theorem in the Plane (cont'd.)

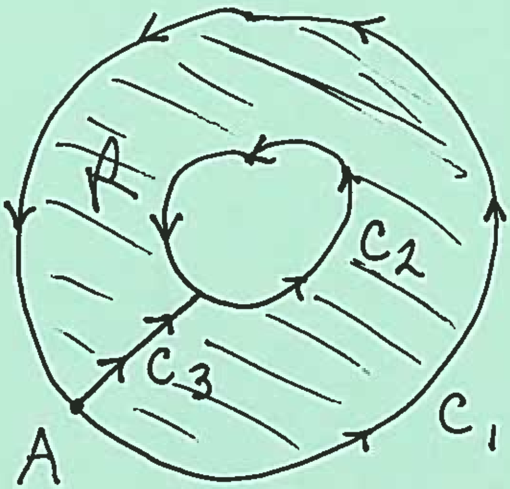
Theorem 3: Green's Theorem

(TWO CURVES Flux – Divergence Normal Form) – Let $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ be a vector field defined on a region R, which is bounded by two simple closed curves C_1 and C_2 (both of which are mapped counter-clockwise). Then



$$\oint_{C_1} \vec{F}\cdot\vec{n}\, ds - \oint_{C_2} \vec{F}\cdot\vec{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$$

proof: (SEE diagram on next page.)



Start at point A and consider the following path C :

$C_1 \to C_3 \to -C_2 \to -C_3$

(ending at point A).

By Theorem 1 (Green's Theorem – normal Form)

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \oint_C \vec{F}\cdot\vec{n}\, ds$$

$$= \oint_{C_1} \vec{F}\cdot\vec{n}\, ds + \int_{C_3} \vec{F}\cdot\vec{n}\, ds + \oint_{-C_2} \vec{F}\cdot\vec{n}\, ds + \int_{-C_3} \vec{F}\cdot\vec{n}\, ds$$

$$= \oint_{C_1} \vec{F}\cdot\vec{n}\, ds + \cancel{\int_{C_3} \vec{F}\cdot\vec{n}\, ds} - \oint_{C_2} \vec{F}\cdot\vec{n}\, ds - \cancel{\int_{C_3} \vec{F}\cdot\vec{n}\, ds}$$

$$= \oint_{C_1} \vec{F}\cdot\vec{n}\, ds - \oint_{C_2} \vec{F}\cdot\vec{n}\, ds$$

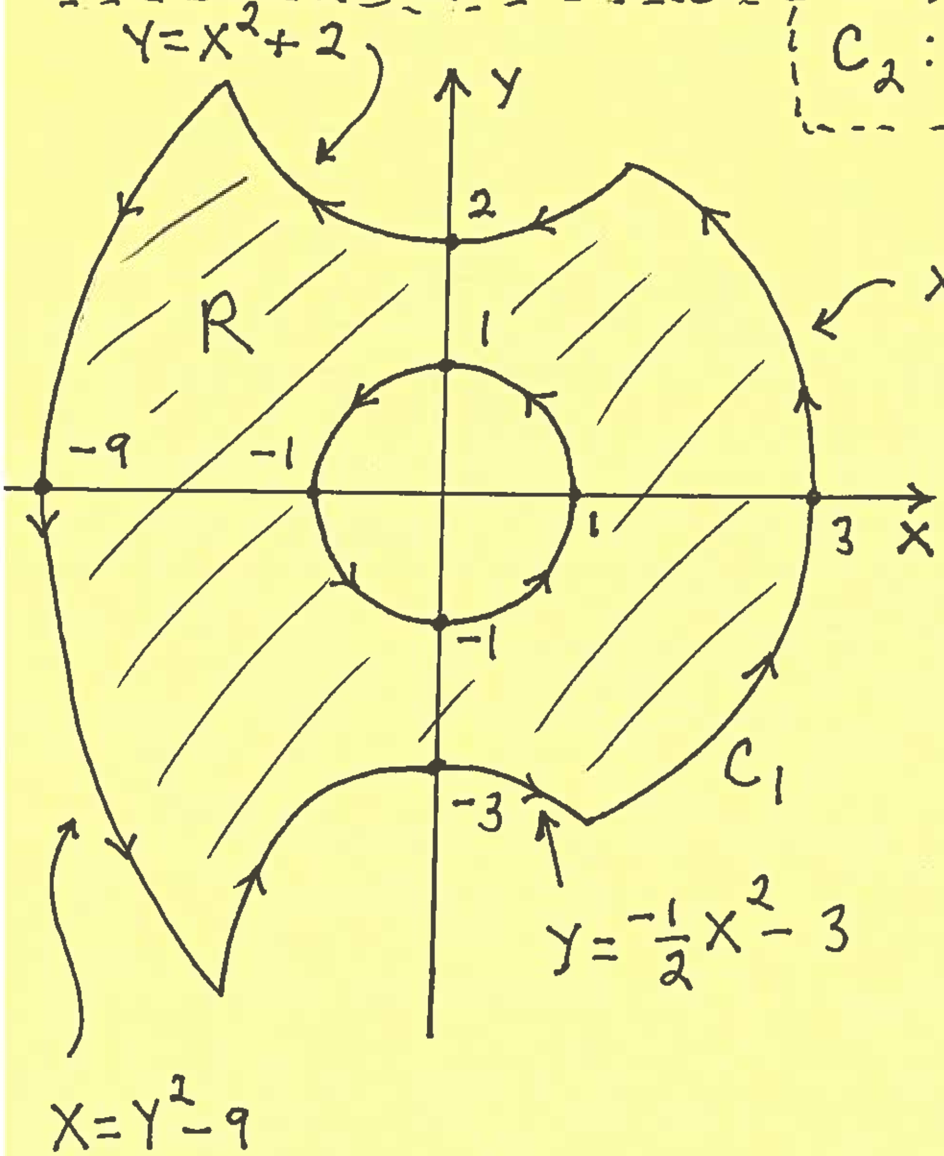
Q.E.D.

Example: Consider the vector field

$$\vec{F}(x,y) = \frac{y}{\sqrt{x^2+y^2}}\,\vec{i} + \frac{-x}{\sqrt{x^2+y^2}}\,\vec{j}$$

and region R enclosed by paths C_1 and C_2 in the diagram below:

$\boxed{C_2:\ x^2+y^2=1}$



1.) Find the Divergence of $\vec{F}$.

2.) Compute Flux $\oint_{C_1} \vec{F}\cdot\vec{n}\,ds$ as easily as possible.

1.) $$M_X = \frac{\sqrt{X^2+Y^2}\cdot(0) - Y\cdot\frac{1}{2}(X^2+Y^2)^{-1/2}(2X)}{\left(\sqrt{X^2+Y^2}\right)^2}$$

$$= \frac{-XY}{(X^2+Y^2)^{3/2}} \quad ;$$

$$N_Y = \frac{\sqrt{X^2+Y^2}\cdot(0) - (-X)\cdot\frac{1}{2}(X^2+Y^2)^{-1/2}(2Y)}{\left(\sqrt{X^2+Y^2}\right)^2}$$

$$= \frac{XY}{(X^2+Y^2)^{3/2}} \,,$$ so that

the Divergence of $\vec{F}$ is

$$\operatorname{div}\vec{F} = \frac{\partial M}{\partial X} + \frac{\partial N}{\partial Y} = 0 \,!$$

2.) Consider path C_2 given by

$$C_2 : \begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad \text{for } 0 \le t \le 2\pi$$

By Theorem 3 (Green's Theorem)

$$\oint_{C_1} \vec{F}\cdot\vec{n}\,ds - \oint_{C_2} \vec{F}\cdot\vec{n}\,ds = \iint_R \underbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)}_{\operatorname{div} \vec{F}} dA$$

$$= \iint_R 0\, dA = 0 \longrightarrow$$

$$\oint_{C_1} \vec{F}\cdot\vec{n}\,ds = \oint_{C_2} \vec{F}\cdot\vec{n}\,ds$$

$$= \oint_{C_2} \left[M \cdot \frac{dy}{dt} - N \cdot \frac{dx}{dt}\right] dt$$

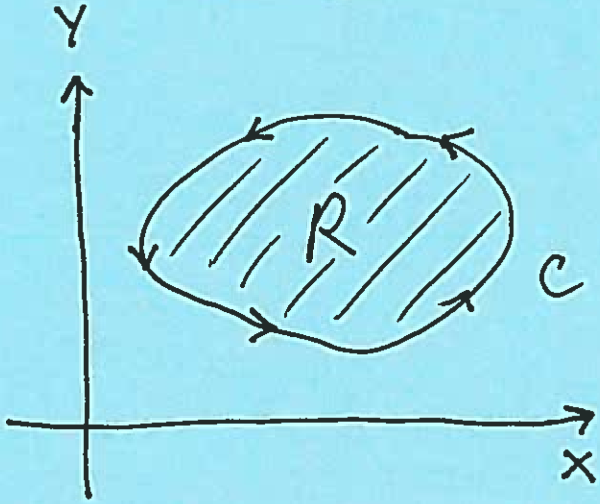
$$= \oint_{C_2} \left[\frac{y}{\sqrt{x^2+y^2}} \cdot \frac{dy}{dt} - \frac{-x}{\sqrt{x^2+y^2}} \cdot \frac{dx}{dt}\right] dt$$

$$= \int_0^{2\pi} \left[\frac{\sin t}{\sqrt{\underbrace{\cos^2 t + \sin^2 t}_{1}}} \cdot (\cos t) + \frac{\cos t}{\sqrt{\underbrace{\cos^2 t + \sin^2 t}_{1}}} \cdot (-\sin t)\right] dt$$

$$= \int_0^{2\pi} (\sin t \cos t - \sin t \cos t)\, dt$$

$$= \int_0^{2\pi} 0\, dt = 0 \;!$$

Finding the Area of an Enclosed Region R by a Closed Loop C Using a Line Integral



Recall:

I.) Area of $R = \iint_R 1 \, dA$

II.) Theorem 1

(Green's Theorem – Normal Form)

$$\oint_C \vec{F} \cdot \vec{n} \, ds = \oint_C M \, dy - N \, dx$$

$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial Y} \right) dA$$

We can conclude that if the Divergence of $\vec{F}$ is

$$\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 1 \text{, then}$$

$$\boxed{\text{area of } R = \iint_R 1\, dA = \oint M\, dy - N\, dx} \; .$$

A convenient vector field to use is

$$\vec{F}(x,y) = \left(\tfrac{1}{2}x\right)\vec{i} + \left(\tfrac{1}{2}y\right)\vec{j} ,$$

so that

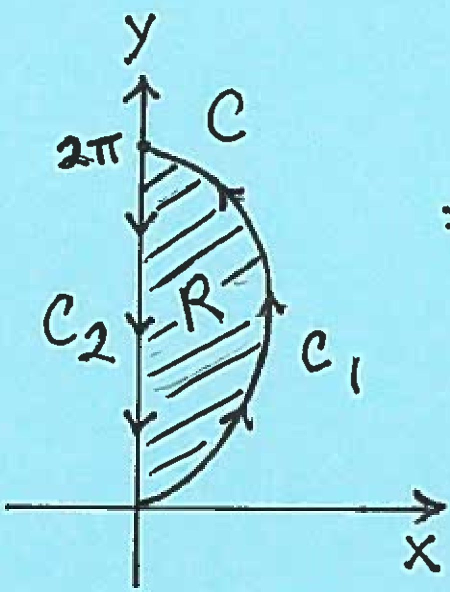
$$\begin{aligned} \text{area of } R &= \oint M\, dy - N\, dx \\ &= \oint \left(\tfrac{1}{2}x\right) dy - \left(\tfrac{1}{2}y\right) dx \\ &= \frac{1}{2} \oint x\, dy - y\, dx \, , \quad \text{i.e.,} \end{aligned}$$

$$\boxed{\text{area of } R = \frac{1}{2} \oint x\, dy - y\, dx} \; .$$

<u>Example</u>: Use a Line Integral to find the area of region R enclosed by path C given by

$$C \begin{cases} C_1 : \begin{cases} x = 1 - \cos t \\ y = t - \sin t \end{cases} \text{ for } 0 \le t \le 2\pi \\ C_2 : \begin{cases} x = 0 \\ y = 2\pi - t \end{cases} \text{ for } 0 \le t \le 2\pi \end{cases}$$

(SEE the next page.)



$$\oint_C x\,dy - y\,dx$$

$$= \int_{C_1} x\,dy - y\,dx + \int_{C_2} x\,dy - y\,dx \;;$$

$$\int_{C_2} x\,dy - y\,dx = \int_0^{2\pi} \left[x \cdot \frac{dy}{dt} - y \cdot \frac{dx}{dt}\right] dt$$

$$= \int_0^{2\pi} \left[(0)(-1) - (2\pi - t)(0)\right] dt = \int_0^{2\pi} 0 \, dt = 0 \;;$$

$$\int_{C_1} x\,dy - y\,dx = \int_0^{2\pi} \left[x \cdot \frac{dy}{dt} - y \cdot \frac{dx}{dt}\right] dt$$

$$= \int_0^{2\pi} \left[(1-\cos t)(1-\cos t) - (t - \sin t)(\sin t)\right] dt$$

$$= \int_0^{2\pi} \left[1 - 2\cos t + \underline{\cos^2 t} - t\sin t + \underline{\underline{\sin^2 t}}\right] dt$$

$$= \int_0^{2\pi} \left[2 - 2\cos t - t\sin t\right] dt$$

$$= (2t - 2\sin t)\Big|_0^{2\pi} - \int_0^{2\pi} t \sin t \, dt$$

(Let $u = t$, $dv = \sin t\, dt$

$\rightarrow du = 1\, dt$, $v = -\cos t$

and $t: 0 \rightarrow 2\pi$ so $u: 0 \rightarrow 2\pi$)

$$= (4\pi - 2\,\overset{0}{\cancel{\sin 2\pi}}) - (0 - 2\,\overset{0}{\cancel{\sin 0}})$$

$$- \left\{ -t\cos t \Big|_0^{2\pi} - - \int_0^{2\pi} \cos t \, dt \right\}$$

$$= 4\pi - \left\{ -(2\pi\,\overset{1}{\cancel{\cos 2\pi}} - 0) + \sin t \Big|_0^{2\pi} \right.$$

$$= 4\pi + 2\pi + (\overset{0}{\cancel{\sin 2\pi}} - \overset{0}{\cancel{\sin 0}}) = 6\pi,$$

so that

$$\text{Area of } R = \frac{1}{2} \oint_C x\,dy - y\,dx$$

$$= \frac{1}{2}(6\pi)$$

$$= 3\pi \, .$$