Section 16.4
Thomas Calculus
11th Ed.

Green's Theorem in the Plane

These theorems will make a precise connection between Line Integrals (for work, circulation, and flux) on closed curves C and Double Integrals on the region R enclosed by path C.

Definition: Consider the vector field $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$.

The Divergence of $\vec{F}$ at the point (x,y) in R is the scalar function

$$\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} .$$

Example: Find the Divergence for each vector field at the given points.

1.) $\vec{F}(x,y) = (x^2 y)\vec{i} + (x - y^2)\vec{j}$

a.) $(-1, 3)$ b.) $(2,1)$ c.) $(1,-1)$

$\text{div } \vec{F} = M_x + N_y = 2xy - 2y$, then

a.) $\text{div } \vec{F}(-1,3) = 2(-1)(3) - 2(3) = -12$,

b.) $\text{div } \vec{F}(2,1) = 2(2)(1) - 2(1) = 2$,

c.) $\text{div } \vec{F}(1,-1) = 2(1)(-1) - 2(-1) = 0$

2.) $\vec{F}(x,y) = (xe^{y})\vec{i} + (e^{-xy})\vec{j}$

a.) $(0,0)$ b.) $(1,-1)$ c.) $(-2, \ln 3)$

$\text{div } \vec{F} = M_x + N_y = e^{y} - xe^{-xy}$, then

a.) $\text{div } \vec{F}(0,0) = e^{0} - (0)e^{0} = 1$,

b.) $\text{div } \vec{F}(1,-1) = e^{-1} - (1)e^{-(1)(-1)} = \frac{1}{e} - e$,

c.) $\text{div } \vec{F}(-2, \ln 3) = e^{\ln 3} - (-2)e^{-(-2)(\ln 3)}$

$= 3 + 2e^{2\ln 3} = 3 + 2e^{\ln 3^2}$

$= 3 + 2(9) = 21$

Remark: Divergence can be shown to be a measure of expansion

(+ divergence) or compression (− divergence) of a gas or fluid at point (X,Y) in region R.

<u>Definition</u>: Consider the vector field $\vec{F}(X,Y) = M(X,Y)\,\vec{i} + N(X,Y)\,\vec{j}$. The <u>$\vec{k}$-component</u> <u>of</u> <u>curl</u> <u>of</u> <u>$\vec{F}$</u> is denoted by

$$(\text{curl } \vec{F}) \cdot \vec{k}$$

and has the formula

$$(\text{curl } \vec{F}) \cdot \vec{k} = \frac{\partial N}{\partial X} - \frac{\partial M}{\partial Y}$$

<u>Example</u>: Find the k-component of curl for the vector fields at the given points.

1.) $\vec{F}(X,Y) = (X-Y)\,\vec{i} + (Y^2 - X^2)\,\vec{j}$

a.) $(0,0)$ b.) $(1,1)$ c.) $\left(\frac{1}{2}, \frac{1}{2}\right)$

$(\text{curl } \vec{F}) \cdot \vec{k} = N_X - M_Y = -2X - (-1) = 1 - 2X$

a.) $(\text{curl } \vec{F}) \cdot \vec{k}\,(0,0) = 1 - 2(0) = 1$,

b.) $(\text{curl } \vec{F}) \cdot \vec{k}\,(1,1) = 1 - 2(1) = -1$, c.) $(\text{curl}\,\vec{F})\cdot k\left(\frac{1}{2},\frac{1}{2}\right) = 1 - 2\left(\frac{1}{2}\right) = 0$

2.) $\vec{F}(x,y) = (y\sin x)\,\vec{i} + (x\cos y)\,\vec{j}$

a.) $(0,0)$ b.) $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$ c.) $\left(\frac{\pi}{4},\frac{\pi}{4}\right)$

$(\text{curl}\,\vec{F})\cdot\vec{k} = N_x - M_y = \cos y - \sin x$

a.) $(\text{curl}\,\vec{F})\cdot\vec{k}\,(0,0) = \cos(0) - \sin(0) = 1 - 0 = 1,$

b.) $(\text{curl}\,\vec{F})\cdot\vec{k}\left(\frac{\pi}{2},\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) = 0 - 1 = -1,$

c.) $(\text{curl}\,\vec{F})\cdot\vec{k}\left(\frac{\pi}{4},\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 0$

Remark: It can be shown that the $\vec{k}$-component of curl of $\vec{F}$ at point (x,y) is a measure of counter-clockwise fluid circulation (+ value) or clockwise fluid circulation (− value) at the point (x,y) in region R.

Recall: $\text{Flux} = \oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx$

Theorem 1: Green's Theorem (Flux-Divergence or Normal Form) –

Let $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ be a defined on a region R enclosed by a simple closed curve C. Then the Flux of $\vec{F}$ across C is

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \underbrace{\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)}_{\operatorname{div}\vec{F}} dA$$

Example: Verify Theorem 1 for $\vec{F}(x,y) = (3x)\vec{i} + (2y)\vec{j}$ and closed path C: circle $x^2+y^2=4$.

Solution: $C: \begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$ for $0 \le t \le 2\pi$

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx$$

$$= \int_0^{2\pi} \left[(3x)\left(\frac{dy}{dt}\right) - (2y)\left(\frac{dx}{dt}\right)\right] dt$$

$$= \int_0^{2\pi} \left[(6\cos t)(2\cos t) - (4\sin t)(-2\sin t)\right] dt$$

$$= \int_0^{2\pi} \left[12\cos^2 t + 8\sin^2 t\right] dt$$

$$= \int_0^{2\pi} [4\cos^2 t + 8(\underbrace{\cos^2 t + \sin^2 t}_{1})]\, dt$$

$$= \int_0^{2\pi} \left[4 \cdot \tfrac{1}{2}(1+\cos 2t) + 8\right] dt$$

$$= \left[2\left(t + \tfrac{1}{2}\sin 2t\right) + 8t\right]\Big|_0^{2\pi}$$

$$= \left[10t + \sin 2t\right]\Big|_0^{2\pi}$$

$$= (20\pi + \cancelto{0}{\sin(4\pi)}) - (0 + \cancelto{0}{\sin(0)})$$

$$= \boxed{20\pi} \quad ; \quad \text{and}$$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \iint_R (3+2)\, dA$$

$$= 5 \iint_R 1\, dA = 5 \cdot (\text{Area } R)$$

$$= 5(\pi(2)^2) = \boxed{20\pi} \,.$$

<u>Recall</u>: Work, Flow, Circulation is

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C M\, dx + N\, dy \,.$$

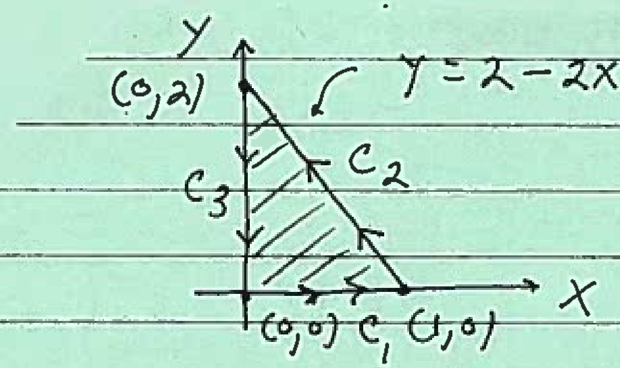
Theorem 2 : Green's Theorem (Circulation – Curl or Tangential Form) – Let $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ be a vector field defined on a region R enclosed by a simple closed curve C. Then the counter-clockwise circulation of $\vec{F}$ on path C is

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{(\text{curl}\,\vec{F})\cdot\vec{k}} dA$$

Example : Verify Theorem 2 for $\vec{F}(x,y) = (xy)\,\vec{i} + (x-y)\,\vec{j}$ and path

C: line segment from (0,0) to (1,0), then line segment from (1,0) to (0,2), then line segment from (0,2) to (0,0).

Solution :



$$\oint_C \vec{F}\cdot\vec{T}\,ds$$

$$= \oint_C M\,dx + N\,dy$$

$C_1 : \begin{cases} x = t \\ y = 0 \end{cases}$ for $0 \le t \le 1$

$$= \int_{C_1} M\,dx + N\,\cancelto{0}{dy}$$

$C_2 : \begin{cases} x = 1-t \\ y = 2t \end{cases}$ for $0 \le t \le 1$

$$+ \int_{C_2} M\,dx + N\,dy$$

$C_3 : \begin{cases} x = 0 \\ y = 2-t \end{cases}$ for $0 \le t \le 2$

$$+ \int_{C_3} M\,\cancelto{0}{dx} + N\,dy$$

$$= \int_0^1 x\cancelto{0}{y}\,dx + \int_0^1 \left[(xy)\frac{dx}{dt} + (x-y)\frac{dy}{dt}\right] dt$$

$$+ \int_0^2 (\cancelto{0}{x} - y)\frac{dy}{dt}\,dt$$

$$= \int_0^1 \left[(1-t)(2t)(-1) + ((1-t)-2t)(2)\right] dt$$

$$\cancel{+ \int_0^2 (t-2)(-1)\,dt}$$

$$= \int_0^1 [2t^2 - 2t + 2 - 6t]\, dt + \int_0^2 (2-t)\, dt$$

$$= \int_0^1 (2t^2 - 8t + 2)\, dt + \int_0^2 (2-t)\, dt$$

$$= \left(\frac{2}{3}t^3 - 4t^2 + 2t\right)\Big|_0^1 + \left(2t - \frac{1}{2}t^2\right)\Big|_0^2$$

$$= \frac{2}{3} - \cancel{4} + \cancel{2} + \cancel{4} - \cancel{2} = \boxed{\frac{2}{3}};$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R (1-x)\, dA$$

$$= \int_0^1 \int_0^{2-2x} (1-x)\, dy\, dx$$

$$= \int_0^1 (1-x)\,y \Big|_{y=0}^{y=2-2x} dx$$

$$= \int_0^1 (1-x)(2-2x)\, dx$$

$$= \int_0^1 (2 - 2x - 2x + 2x^2)\, dx$$

$$= \int_0^1 (2 - 4x + 2x^2)\, dx$$

$$= \left(2x - 2x^2 + \frac{2}{3}x^3\right)\Big|_0^1$$

$$= 2 - 2 + 2/3$$

$$= \boxed{2/3}$$