Section 15.2
Thomas Calculus
11th Ed.

Moments, Center of Mass, Centroid, Mass, Moment of Inertia

Recall: Moment in One Dimension:

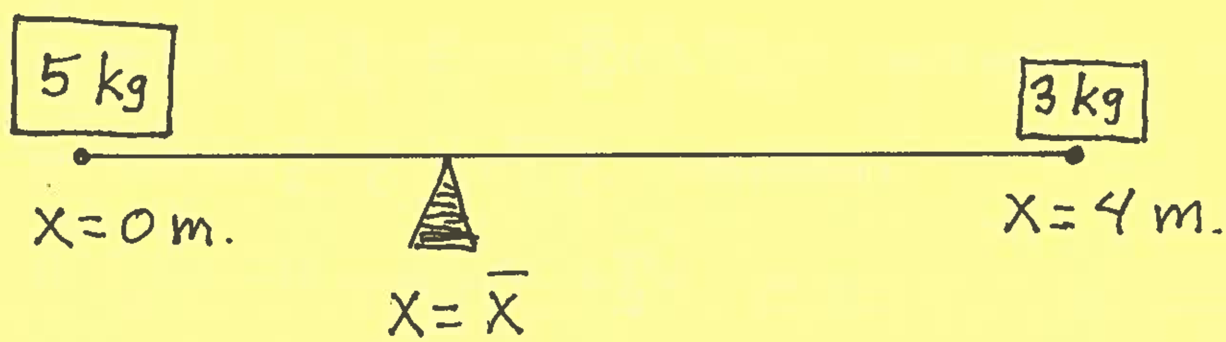
Definition: Consider the system of masses $m_1, m_2, \ldots, m_n$ positioned on an axis at points $x_1, x_2, \ldots, x_n$, resp. The moment of this system about $x=\overline{x}$ is

$$\sum_{i=1}^{n} m_i\,(x_i-\overline{x}) = m_1(x_1-\overline{x})+m_2(x_2-\overline{x}) + \cdots + m_n(x_n-\overline{x})$$

mass

signed distances (since some may be "+" and some may be "−")

Example: Find the total moment of the following system about $x = \bar{x}$. Then find the $\bar{x}$ that "balances" the system.



The Moment about $x = \bar{x}$ is

$$M_{x=\bar{x}} = (5)(0-\bar{x}) + (3)(4-\bar{x})$$
$$= -5\bar{x} + 12 - 3\bar{x}$$
$$= 12 - 8\bar{x} \quad (\text{kg})(\text{m.});$$

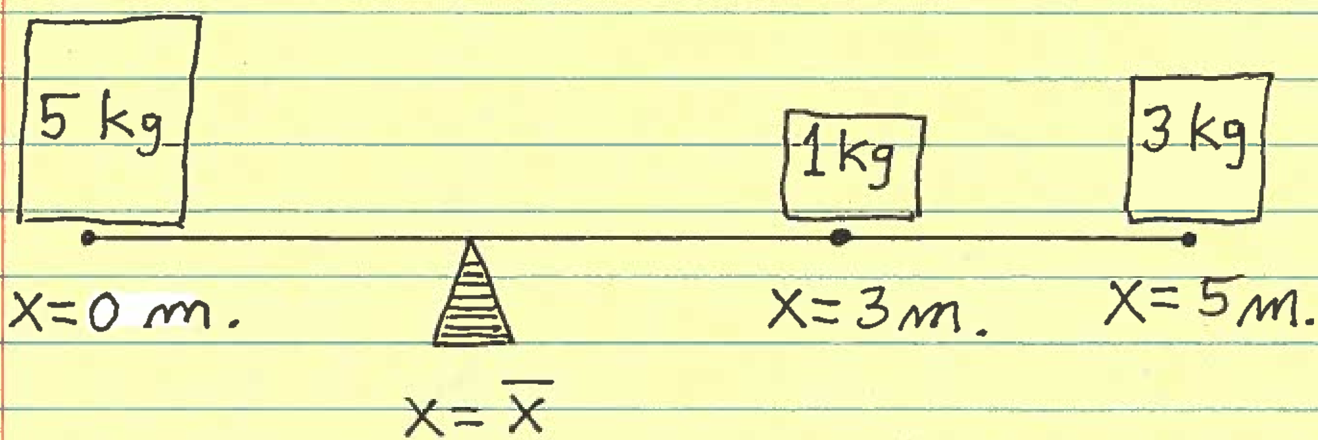
The system "balances" when

$$M_{x=\bar{x}} = 0 \longrightarrow 12 - 8\bar{x} = 0 \longrightarrow$$

$$8\bar{x} = 12 \longrightarrow \bar{x} = \frac{12}{8} = \frac{3}{2}, \text{ i.e.,}$$

$$\boxed{\bar{x} = \frac{3}{2} \text{ m.}}$$

Example: Find the point $x = \overline{x}$ that will "balance" the given system of masses and positions.



The MOMENT of this system about $x = \overline{x}$ is

$$M_{x=\overline{x}} = (5)(0-\overline{x}) + (1)(3-\overline{x}) + (3)(5-\overline{x})$$
$$= -5\overline{x} + 3 - \overline{x} + 15 - 3\overline{x}$$
$$= 18 - 9\overline{x} \quad ;$$

at the "balance" point the moment must be ZERO, i.e.,

$$M_{x=\overline{x}} = 18 - 9\overline{x} = 0 \longrightarrow$$

$$\boxed{\overline{x} = 2 \text{ m.}}$$

Let's generalize and apply the concept of MOMENT to a TWO-DIMENSIONAL SYSTEM, e.g., a flat plate of variable density.

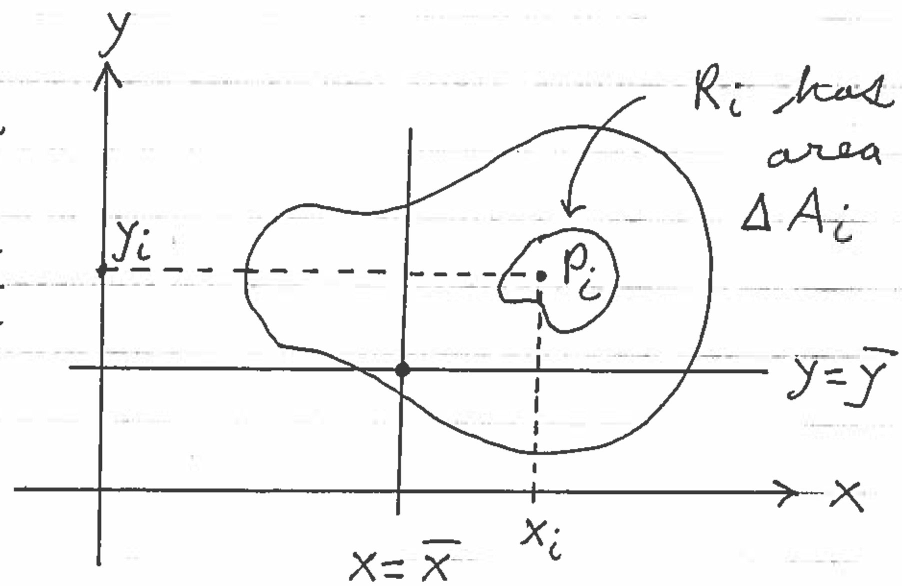
SEE the following 4-page handout.

Math 21D
Kouba
Moments, Center of Mass, Centroid, and Moment of Inertia in 2D-Space

Consider a flat region R with variable density $\left(\frac{\text{mass}}{\text{area}}\text{ units}\right)$ $\delta(P)$ at point P. Partition region R into n parts $R_1, R_2, R_3, \dots, R_n$ of areas $\Delta A_1, \Delta A_2, \Delta A_3, \dots, \Delta A_n$, resp. Pick sampling point

$$P_i = (x_i, y_i) \text{ in } R_i \text{ for } i = 1, 2, 3, \dots, n.$$

Consider the vertical line $x = \bar{x}$. The moment of R_i about line $x = \bar{x}$ is



$$M_i = (\text{mass})(\text{distance})$$
$$= (\text{area})(\text{density})(\text{distance})$$
$$= (\Delta A_i)(\delta(P_i))(x_i - \bar{x}).$$

The <u>total moment of region R</u>

about line $x=\bar{x}$ is

$$M_{x=\bar{x}} = \lim_{mesh\to 0} \sum_{i=1}^{n} \delta(P_i)\cdot(x_i-\bar{x})\cdot\Delta A_i \rightarrow$$

$$\boxed{M_{x=\bar{x}} = \iint_R \delta(P)(x-\bar{x})\,dA} .$$

at the center of mass of R we have

$M_{x=\bar{x}} = 0 \rightarrow$

$$\iint_R \delta(P)(x-\bar{x})\,dA = 0 \rightarrow$$

$$\iint_R x\cdot\delta(P)\,dA - \iint_R \bar{x}\cdot\delta(P)\,dA = 0 \rightarrow$$

$$\iint_R x\cdot\delta(P)\,dA = \bar{x}\iint_R \delta(P)\,dA \rightarrow$$

$$\boxed{\bar{x} = \frac{\iint_R x\cdot\delta(P)\,dA}{\iint_R \delta(P)\,dA}} .$$

Consider the horizontal line $y=\bar{y}$. In an analagous fashion, the total moment of region R about line $y=\bar{y}$ is

$$\boxed{M_{y=\bar{y}} = \iint_R \delta(P)(y-\bar{y})\,dA} .$$

at the <u>center of mass</u> of R we have

$M_{y=\bar{y}} = 0 \longrightarrow$

$$\boxed{\bar{y} = \frac{\iint_R y\,\delta(P)\,dA}{\iint_R \delta(P)\,dA}}.$$

<u>Note</u>: $\iint_R \delta(P)\,dA$ = total mass of R

<u>Note</u>: If the density at the point P is $\delta(P) = k$, a constant, then the center of mass is called the <u>centroid</u> (geometric center) of region R and is given by

$$\boxed{\bar{x} = \frac{\iint_R x\,dA}{\iint_R 1\,dA}}$$

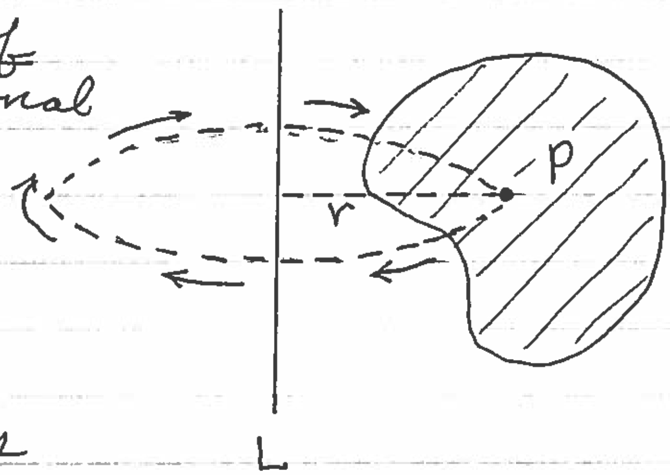
and

$$\boxed{\bar{y} = \frac{\iint_R y\,dA}{\iint_R 1\,dA}}.$$

Note : $\iint_R 1\, dA$ = total area of R

Consider a flat region R with density $\delta(P)$ at point P and which is rotating around a line L or fixed point P_0.

The moment of inertia (rotational inertia) is a measure of the object's resistance to changes in its rotation. It depends on the object's total mass and how far each bit of mass is from the axis or point of rotation.



Def: The moment of inertia of R is

$$\boxed{\text{M. of I.} = \iint_R r^2\, \delta(P)\, dA},$$

where r is the distance from P to the axis or point of rotation.

<u>Example</u>: Consider plate R of density $\delta(x,y) = e^{x+y}$ gm./cm.^{2} at point $P=(x,y)$ and which is bounded by the graphs of $y = 2 - \frac{1}{2}x$, $x=0$, and $y=0$.

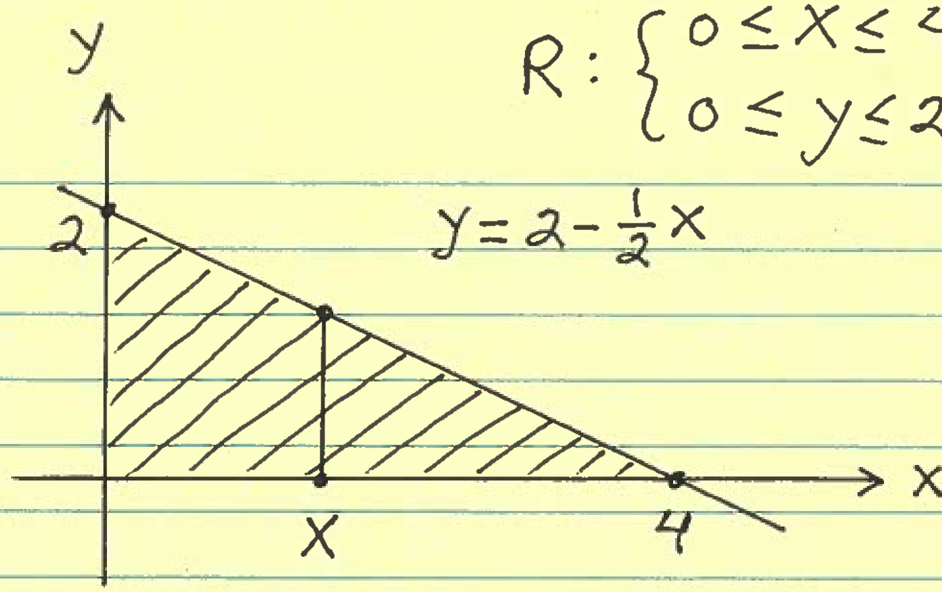
SET UP but DO NOT EVALUATE Double Integrals representing the plate's

1.) Area.
2.) Mass.
3.) Center of Mass.
4.) Centroid.
5.) Moment about the
    a.) x-axis.
    b.) y-axis.
    c.) line $y=3$.
    d.) line $x=-2$.
6.) Moment of Inertia about the
    a.) point $(3,2)$.
    b.) x-axis.
    c.) y-axis.

$$R: \begin{cases} 0 \le x \le 4 \\ 0 \le y \le 2 - \frac{1}{2}x \end{cases}$$

density

$\delta(P) = e^{x+y}$



1.) $\text{Area} = \int_0^4 \int_0^{2-\frac{1}{2}x} 1 \, dy \, dx \quad \text{cm.}^2$

2.) $\text{Mass} = \int_0^4 \int_0^{2-\frac{1}{2}x} e^{x+y} \, dy \, dx \quad \text{gm.}$

3.)

$$\bar{x} = \frac{\int_0^4 \int_0^{2-\frac{1}{2}x} x e^{x+y} \, dy \, dx}{\int_0^4 \int_0^{2-\frac{1}{2}x} e^{x+y} \, dy \, dx} \quad \text{cm.},$$

$$\bar{y} = \frac{\int_0^4 \int_0^{2-\frac{1}{2}x} y e^{x+y} \, dy \, dx}{\int_0^4 \int_0^{2-\frac{1}{2}x} e^{x+y} \, dy \, dx} \quad \text{cm.}$$

4.) $$\bar{X} = \frac{\int_0^4 \int_0^{2-\frac{1}{2}X} x \, dy \, dx}{\int_0^4 \int_0^{2-\frac{1}{2}X} 1 \, dy \, dx} \text{ cm.},$$

$$\bar{Y} = \frac{\int_0^4 \int_0^{2-\frac{1}{2}X} y \, dy \, dx}{\int_0^4 \int_0^{2-\frac{1}{2}X} 1 \, dy \, dx} \text{ cm.}$$

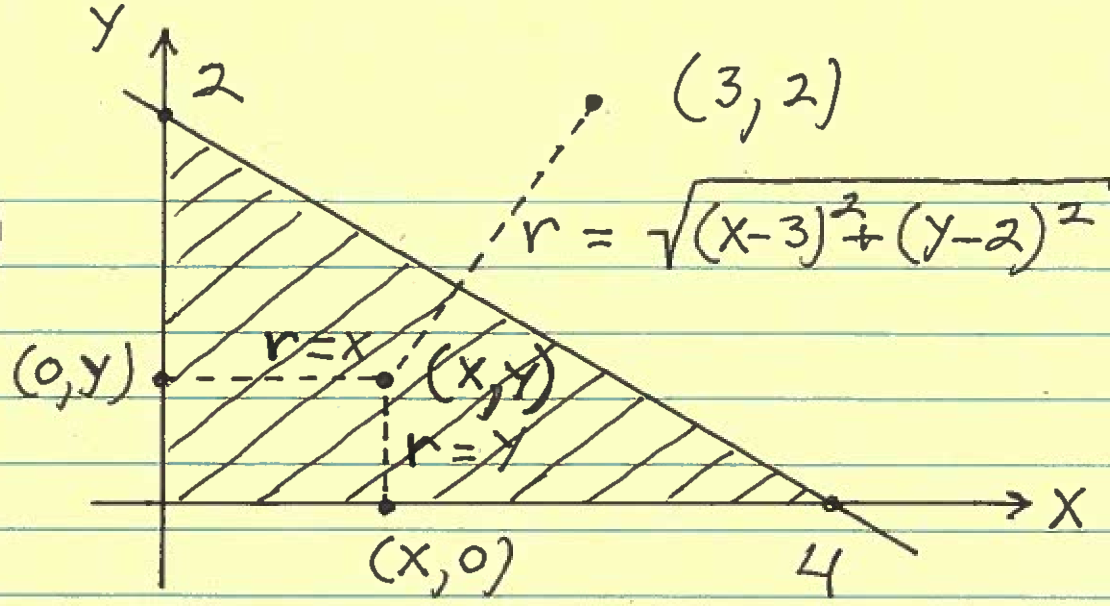
5.) a.) $$M_{Y=0} = \int_0^4 \int_0^{2-\frac{1}{2}X} (Y-0) e^{X+Y} \, dy \, dX \text{ (gm.)(cm.)}$$

b.) $$M_{X=0} = \int_0^4 \int_0^{2-\frac{1}{2}X} (X-0) e^{X+Y} \, dy \, dX \text{ (gm.)(cm.)}$$

c.) $$M_{Y=3} = \int_0^4 \int_0^{2-\frac{1}{2}X} (Y-3) e^{X+Y} \, dy \, dX \text{ (gm.)(cm.)}$$

d.) $$M_{X=-2} = \int_0^4 \int_0^{2-\frac{1}{2}X} (X+2) e^{X+Y} \, dY \, dX \text{ (gm.)(cm.)}$$

6.)



a.) $\text{M. of I.} = \int_0^4 \int_0^{2-\frac{1}{2}x} \left((x-3)^2 + (y-2)^2\right) e^{x+y} \, dy\, dx$

(gm.)(cm.^{2})

b.) $\text{M. of I.} = \int_0^4 \int_0^{2-\frac{1}{2}x} y^2 e^{x+y} \, dy\, dx$

(gm.)(cm.^{2})

c.) $\text{M. of I.} = \int_0^4 \int_0^{2-\frac{1}{2}x} x^2 e^{x+y} \, dy\, dx$

(gm.)(cm.^{2})

Example: Consider plate R of density $\delta(x,y) = x^2+y^2+1$ gm./cm.^{3} at point $P=(x,y)$ and which is bounded by the graphs of $y=x^2$ and $y=x+2$. SET UP but do not evaluate Double Integrals representing the

1.) area of R.

2.) Mass of R.

3.) Center of Mass of R, $\bar{x}$ ONLY.

4.) Centroid of R, $\bar{y}$ ONLY.

5.) Moment of R about the

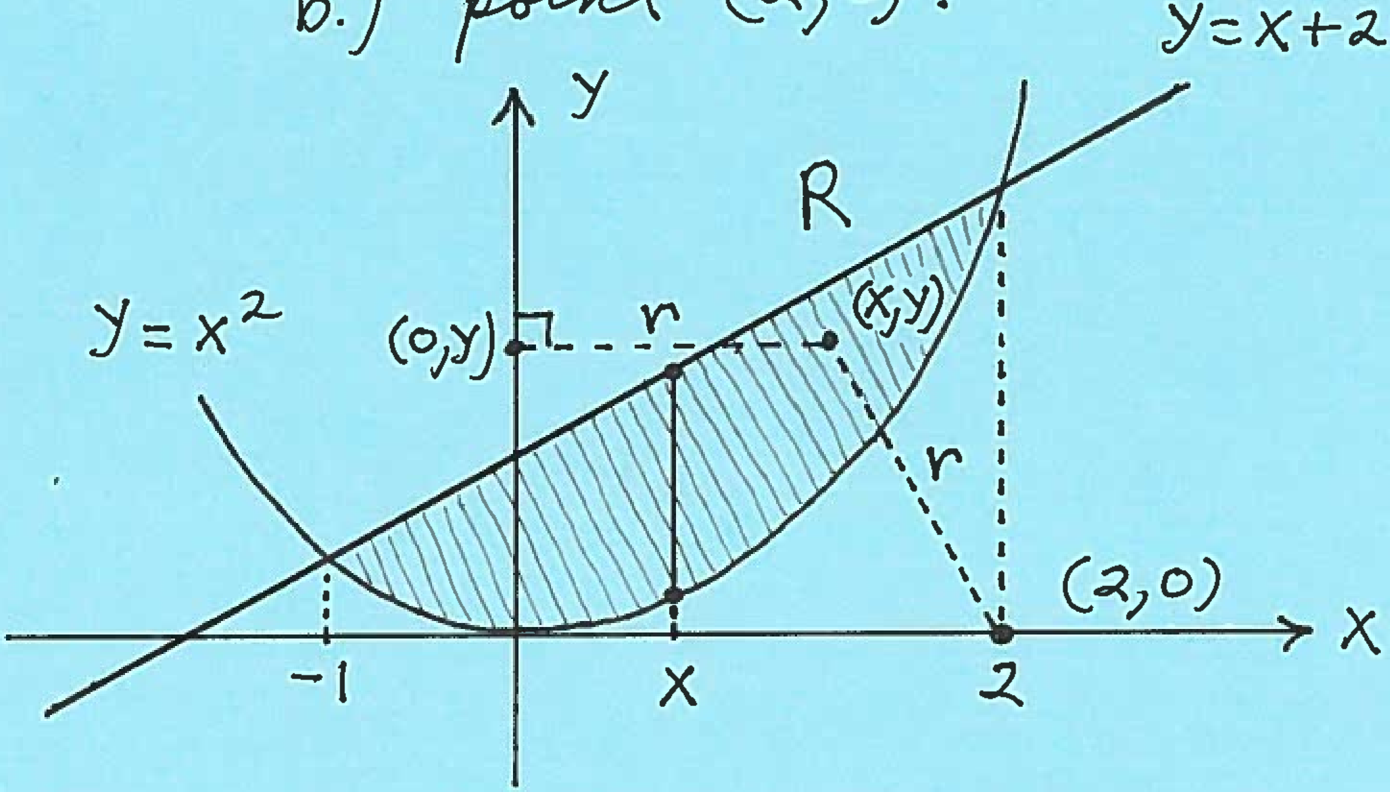
a.) x-axis.

b.) line $x=3$.

6.) Moment of Inertia of R about the

a.) y-axis.

b.) point (2,0).



$x^2 = x+2 \rightarrow$

$x^2 - x - 2 = 0 \rightarrow$

$(x-2)(x+1) = 0 \rightarrow$

$x=2, \; x=-1$

$$R: \begin{cases} -1 \le x \le 2 \\ x^2 \le y \le x+2 \end{cases}$$

1.) $\text{Area} = \int_{-1}^{2} \int_{x^2}^{x+2} 1 \, dy\, dx \ \text{cm.}^2$

2.) $\text{Mass} = \int_{-1}^{2} \int_{x^2}^{x+2} (x^2+y^2+1)\, dy\, dx \ \text{gm.}$

3.)

$$\bar{x} = \frac{\int_{-1}^{2} \int_{x^2}^{x+2} x\cdot(x^2+y^2+1)\, dy\, dx}{\int_{-1}^{2} \int_{x^2}^{x+2} (x^2+y^2+1)\, dy\, dx}$$

4.)

$$\bar{y} = \frac{\int_{-1}^{2} \int_{x^2}^{x+2} y \ dy\, dx}{\int_{-1}^{2} \int_{x^2}^{x+2} 1 \ dy\, dx}$$

5.) a.) $M_{y=0} = \int_{-1}^{2} \int_{x^2}^{x+2} (y-0)(x^2+y^2+1)\, dy\, dx$ (gm.)(cm.)

b.) $M_{x=3} = \int_{-1}^{2} \int_{x^2}^{x+2} (x-3)(x^2+y^2+1)\, dy\, dx$

(gm.)(cm.)

6.) $r = \sqrt{(x-0)^2 + (y-y)^2} = \sqrt{x^2} = |x|$, so

a.)

$\text{M. of I.} = \int_{-1}^{2} \int_{x^2}^{x+2} (|x|)^2 (x^2+y^2+1)\, dy\, dx$

(gm.)(cm.)^2

b.) $r = \sqrt{(x-2)^2 + (y-0)^2} = \sqrt{(x-2)^2 + y^2}$, so

$\text{M. of I.} = \int_{-1}^{2} \int_{x^2}^{x+2} \left(\sqrt{(x-2)^2+y^2}\right)^2 \cdot (x^2+y^2+1) \cdot dy\, dx$

(gm.)(cm.)^2