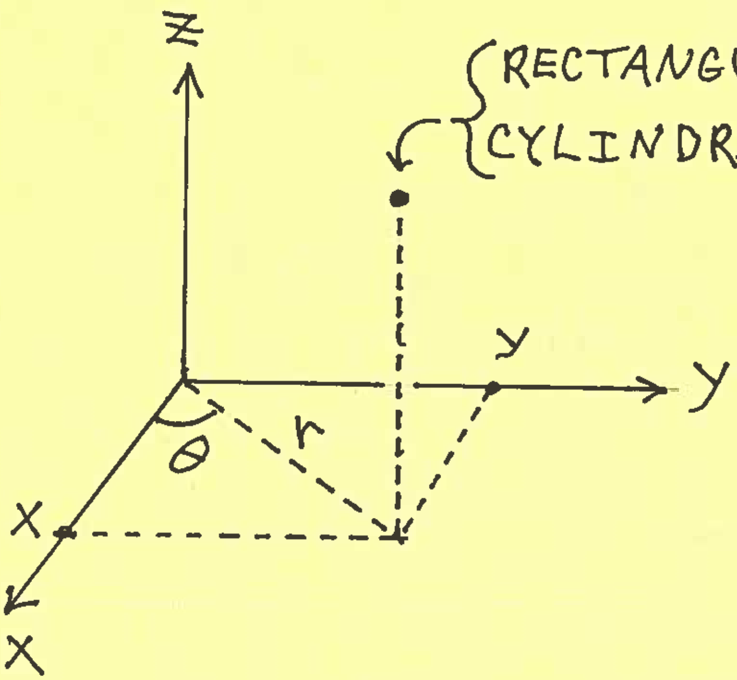
Section 15.6
Thomas Calculus
11 th Ed.

Triple Integrals Over Solid Regions R in 3D-Space Using Cylindrical Coordinates



RECTANGULAR Coord.: (x, y, z)

CYLINDRICAL Coord.: (r, θ, z)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

NOTE: Cylindrical Coordinates are Polar Coordinates in the xy-plane.

Example: Let R be the solid region enclosed by the surfaces $z = x^2 + y^2$ (paraboloid) and

$z = 2y$ (plane).

1.) Sketch the surfaces, their intersection, and the solid R.

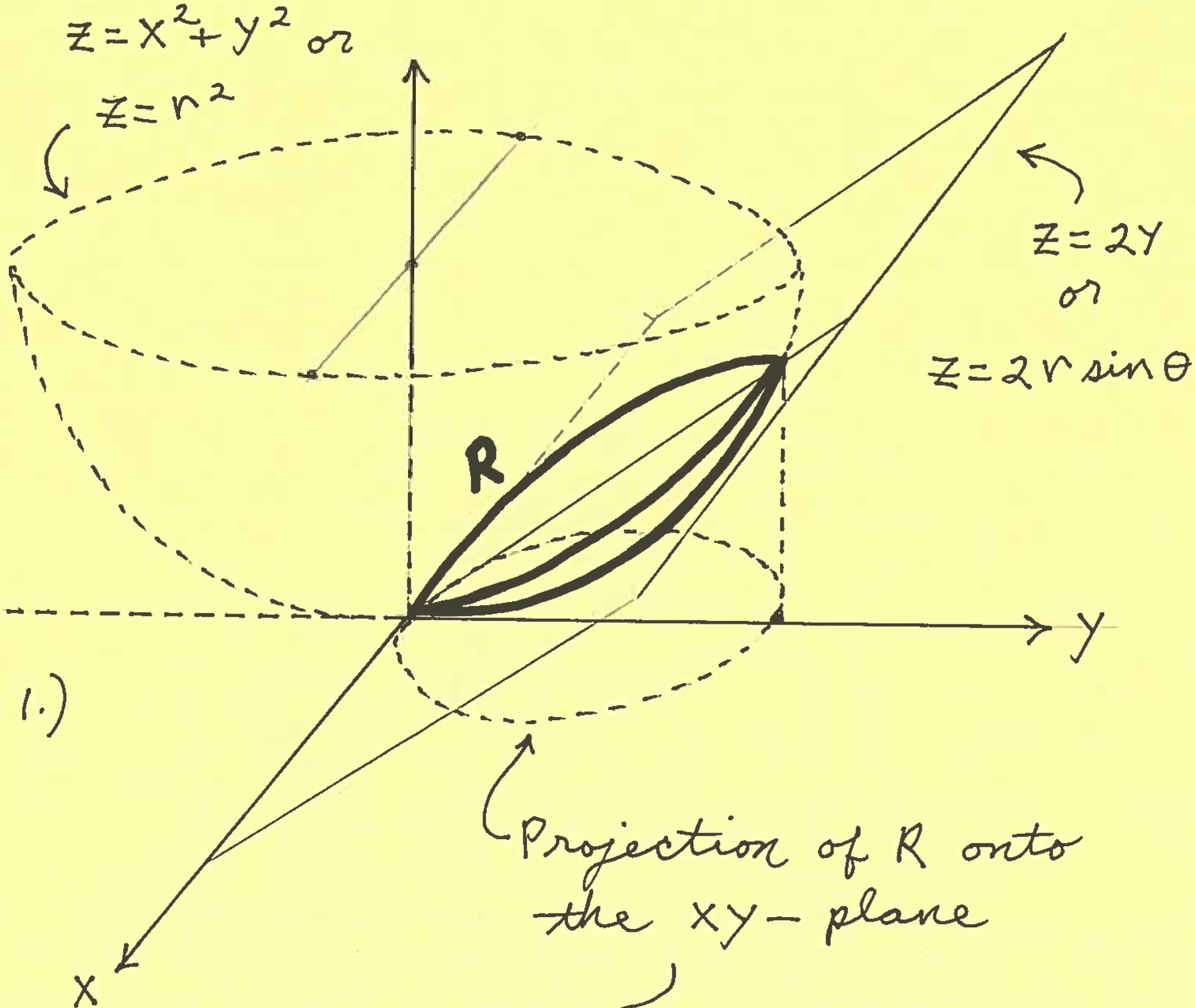
2.) Sketch the PROJECTION of R onto the xy-plane, and describe this projection using Polar Coordinates.

3.) Describe region R using Cylindrical Coordinates.

4.) Find the Volume of R using Cylindrical Coordinates.

5.) Find the Average Value of the function $f(x, y, z) = xy^2z^3$ over region R using Cylindrical Coordinates.

(SET UP ONLY)



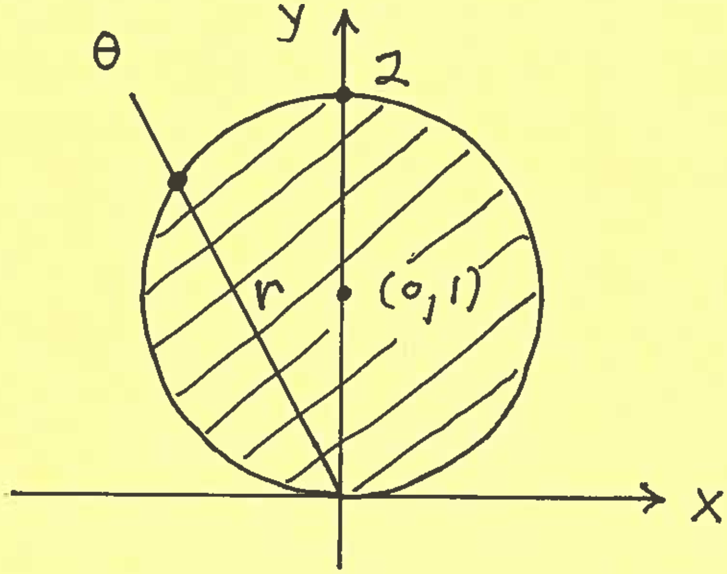
1.)

2.) $x^2 + y^2 = 2y \rightarrow$

$x^2 + y^2 - 2y = 0 \rightarrow$

$x^2 + y^2 - 2y + 1 = 1 \rightarrow$

$x^2 + (y-1)^2 = 1$, a circle of radius $r = 1$ and center $(0, 1)$



$$x^2+(y-1)^2=1$$

or

$$r = 2\sin\theta$$

Projection: $\begin{cases} 0 \le \theta \le \pi \\ 0 \le r \le 2\sin\theta \end{cases}$

3.) $R: \begin{cases} 0 \le \theta \le \pi \\ 0 \le r \le 2\sin\theta \\ r^2 \le z \le 2r\sin\theta \end{cases}$

4.) Volume of $R = \iiint_R 1 \, dV$

$$= \int_0^{\pi} \int_0^{2\sin\theta} \int_{r^2}^{2r\sin\theta} r \, dz \, dr \, d\theta$$

5.) Average Value of $f(P)$ over R is

$$\text{AVE} = \frac{1}{\text{Vol.} R} \iiint_R f(P) \, dV$$

$$= \frac{1}{\text{Vol.} R} \iiint_R x y^2 z^3 \, dV$$

$$= \frac{1}{\int_0^{\pi}\int_0^{2\sin\theta}\int_{r^2}^{2r\sin\theta} r \, dz\,dr\,d\theta} \cdot$$

$$\int_0^{\pi}\int_0^{2\sin\theta}\int_{r^2}^{2r\sin\theta} (r\cos\theta)(r\sin\theta)^2 z^3 \cdot r \, dz\,dr\,d\theta$$

<u>Example</u>: Convert the following Triple Integral to Cylindrical Coordinates and evaluate it.

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{x^2+y^2} \sqrt{16-x^2-y^2} \, dz\,dy\,dx$$

$y = \sqrt{9-x^2}$
(circle)

$$\begin{cases} 0 \le x \le 3 \\ 0 \le y \le \sqrt{9-x^2} \end{cases}$$

or

$$\begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 3 \end{cases}$$

$$= \int_0^{\frac{\pi}{2}} \int_0^3 \int_0^{r^2} \sqrt{16-r^2} \cdot r \; dz \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^3 \sqrt{16-r^2} \cdot r \cdot z \Big|_{z=0}^{z=r^2} \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^3 r^3 \sqrt{16-r^2} \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^3 r^2 \cdot r \sqrt{16-r^2} \, dr \, d\theta$$

(Let $u = 16 - r^2 \xrightarrow{D} du = -2r\,dr$
$\rightarrow -\frac{1}{2} du = r\,dr$; $r^2 = 16 - u$ and
$r: 0 \rightarrow 3$ so $u: 16 \rightarrow 7$.)

$$= \int_0^{\frac{\pi}{2}} \int_{16}^7 -\frac{1}{2}(16-u) u^{1/2} \, du \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_{16}^7 -\frac{1}{2}(16u^{1/2} - u^{3/2}) \, du \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_{16}^7 \left(-8u^{1/2} + \frac{1}{2} u^{3/2}\right) du \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(-8 \cdot \frac{2}{3} u^{3/2} + \frac{1}{2} \cdot \frac{2}{5} u^{5/2}\right) \Big|_{u=16}^{u=7} \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\left(-\frac{16}{3}(7)^{3/2} + \frac{1}{5}(7)^{5/2} \right) - \left(-\frac{16}{3} \underbrace{(16)^{3/2}}_{64} + \frac{1}{5} \underbrace{(16)^{5/2}}_{1024} \right) \right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[-\frac{16}{3}(7)^{3/2} + \frac{1}{5}(7)^{5/2} + \frac{1024}{3} + \frac{1024}{5} \right] d\theta$$

$$= \left[-\frac{16}{3}(7)^{3/2} + \frac{1}{5}(7)^{5/2} + \frac{1024}{3} + \frac{1024}{5} \right] \theta \Big|_0^{\frac{\pi}{2}}$$

$$= \left[-\frac{16}{3}(7)^{3/2} + \frac{1}{5}(7)^{5/2} + \frac{1024}{3} + \frac{1024}{5} \right] \cdot \frac{\pi}{2}$$

Example: Sketch the solid region R whose Volume is given by the following Triple Integral.

$$\int_0^{2\pi} \int_0^2 \int_r^{\sqrt{8-r^2}} r \cdot dz \, dr \, d\theta$$

$z = r = \sqrt{x^2+y^2} \rightarrow z = \sqrt{x^2+y^2}$ (cone);

$z = \sqrt{8-r^2} = \sqrt{8-x^2-y^2}$ (hemisphere)

since $z^2 = 8-x^2-y^2$ or

$x^2+y^2+z^2 = 8$; Find intersection of surfaces $z = r$ and $z = \sqrt{8-r^2} \rightarrow$

$r = \sqrt{8-r^2} \rightarrow r^2 = 8-r^2 \rightarrow 2r^2 = 8 \rightarrow$

$r^2 = 4 \rightarrow r = 2$ (a circle) ; From Triple Integral we have

$$\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \end{cases}$$