Section 15.5
Thomas Calculus
11th Ed.

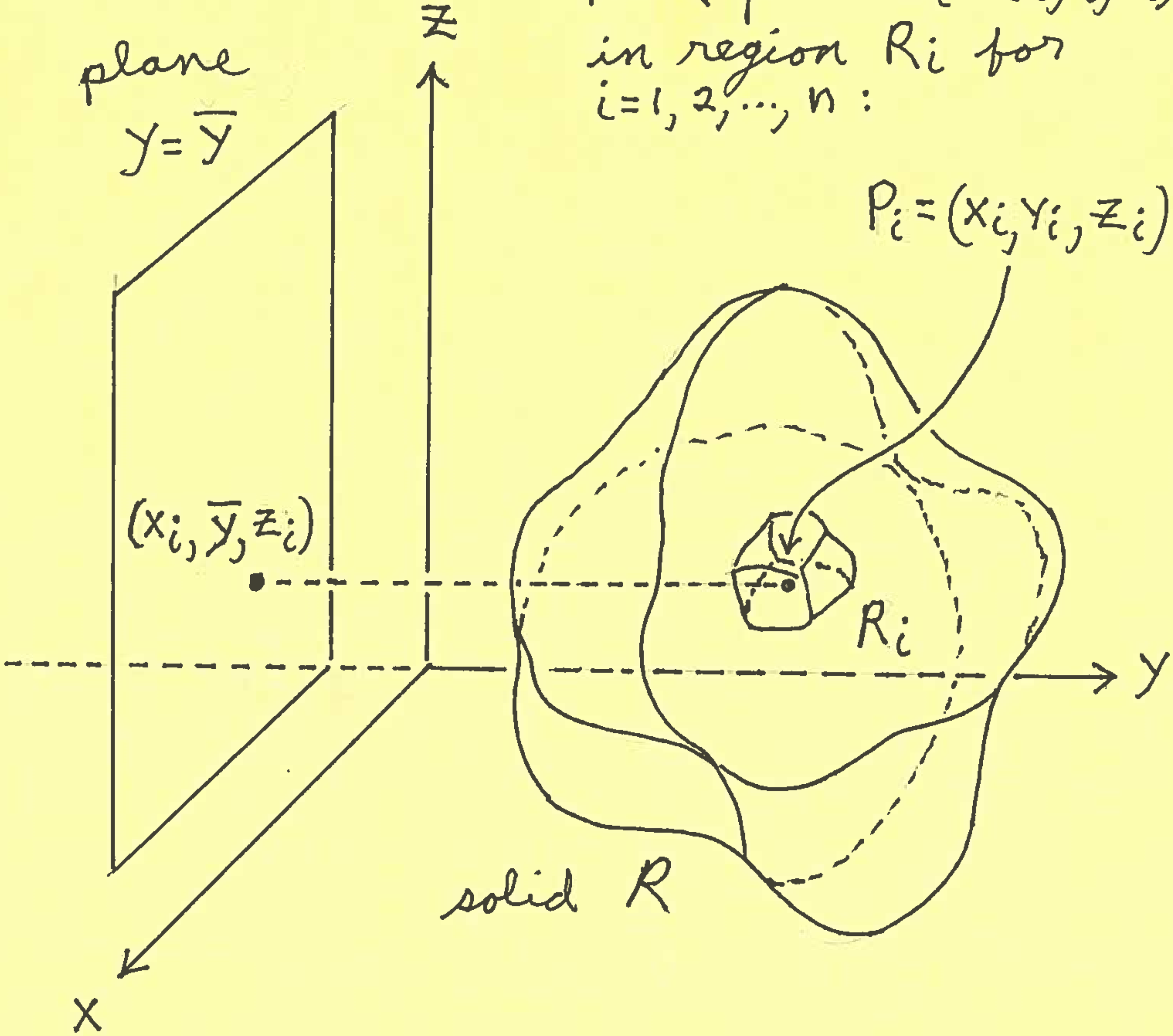
Triple Integrals and Moments in 3D-Space

Consider a solid region R in 3D-Space, where the density at point $P=(x,y,z)$ is $\delta(P)$ $\frac{\text{mass}}{\text{volume}}$ units.

Partition R into n parts $R_1, R_2, \dots, R_n$ of volumes $\Delta V_1, \Delta V_2, \dots, \Delta V_n$ for $i=1,2,\dots,n$.

Let $y=\bar{y}$ be a plane parallel to the xz-plane:

Pick point $P_i = (x_i, y_i, z_i)$ in region R_i for $i = 1, 2, \ldots, n$:



The Moment of solid R_i about the plane $y = \bar{y}$ is

$$(\text{mass})(\text{distance}) = (\text{density})(\text{volume})(\text{distance})$$

$$\approx \delta(P_i) \cdot \Delta V_i \cdot (y_i - \bar{y}),$$

so that the <u>Moment</u> of solid region R <u>about the plane</u> <u>$y = \bar{y}$</u> is

$$M_{y=\bar{y}} = \lim_{\text{mesh}\to 0} \sum_{i=1}^{n} (y_i - \bar{y}) \cdot \delta(P_i) \cdot \Delta V_i \,,$$

i.e.,

$$\boxed{M_{y=\bar{y}} = \iiint_R (y-\bar{y})\, \delta(P)\, dV} \,.$$

Similarly, the <u>Moment</u> of solid region R <u>about the plane $x=\bar{x}$</u> is

$$\boxed{M_{x=\bar{x}} = \iiint_R (x-\bar{x})\, \delta(P)\, dV} \,,$$

and the Moment of solid region R about the plane $z = \bar{z}$ is

$$\boxed{M_{z=\bar{z}} = \iiint_R (z-\bar{z})\, \delta(P)\, dV} \,.$$

To find the Center of Mass $(\bar{x}, \bar{y}, \bar{z})$ of solid region R, set each of these Moments equal to zero, resulting in

$$\boxed{\bar{x} = \frac{\iiint\limits_R x\,\delta(P)\,dV}{\iiint\limits_R \delta(P)\,dV}},$$

$$\boxed{\bar{y} = \frac{\iiint\limits_R y\,\delta(P)\,dV}{\iiint\limits_R \delta(P)\,dV}}, \quad \text{and}$$

$$\boxed{\bar{z} = \frac{\iiint\limits_R z\,\delta(P)\,dV}{\iiint\limits_R \delta(P)\,dV}}$$

If density $\delta(P) = k$ is constant, then the <u>Centroid</u> $(\bar{x}, \bar{y}, \bar{z})$ of solid region R is

$$\boxed{\bar{x} = \frac{\iiint_R x \, dV}{\iiint_R 1 \, dV}},$$

$$\boxed{\bar{y} = \frac{\iiint_R y \, dV}{\iiint_R 1 \, dV}}, \quad \text{and}$$

$$\boxed{\bar{z} = \frac{\iiint_R z \, dV}{\iiint_R 1 \, dV}}.$$

The <u>Moment of Inertia</u> of solid region R about a point P_0 or a line L is

$$\boxed{\text{Moment of Inertia} = \iiint_R r^2 \delta(P)\, dV} \, ,$$

where r is the <u>distance</u> from point $P = (x, y, z)$ in region R to P_0 or L.

<u>Example</u>: Consider the solid region R enclosed by the surfaces $z = 9 - x^2 - y^2$ (a paraboloid) and $z = 5$ (a plane). Assume that density at point $P = (x, y, z)$ in R is given by

$$\delta(P) = x + y^2 + 3z \quad \frac{kg.}{m.^3}$$

1.) Sketch the surfaces, their intersection, and solid R.

2.) Sketch the Projection of R onto the xy-plane and Describe this Projection using

a.) Vertical Cross-Sections.

b.) Horizontal Cross-Sections.

SET UP ONLY Double Integrals or Triple Integrals for the

3.) Area of the Projection.

4.) Volume of R.

5.) Mass of R.

6.) Moment of R about the plane $X = 4$.

7.) Center of Mass of R ($\bar{X}$ ONLY).

8.) Centroid of R ($\bar{Z}$ ONLY).

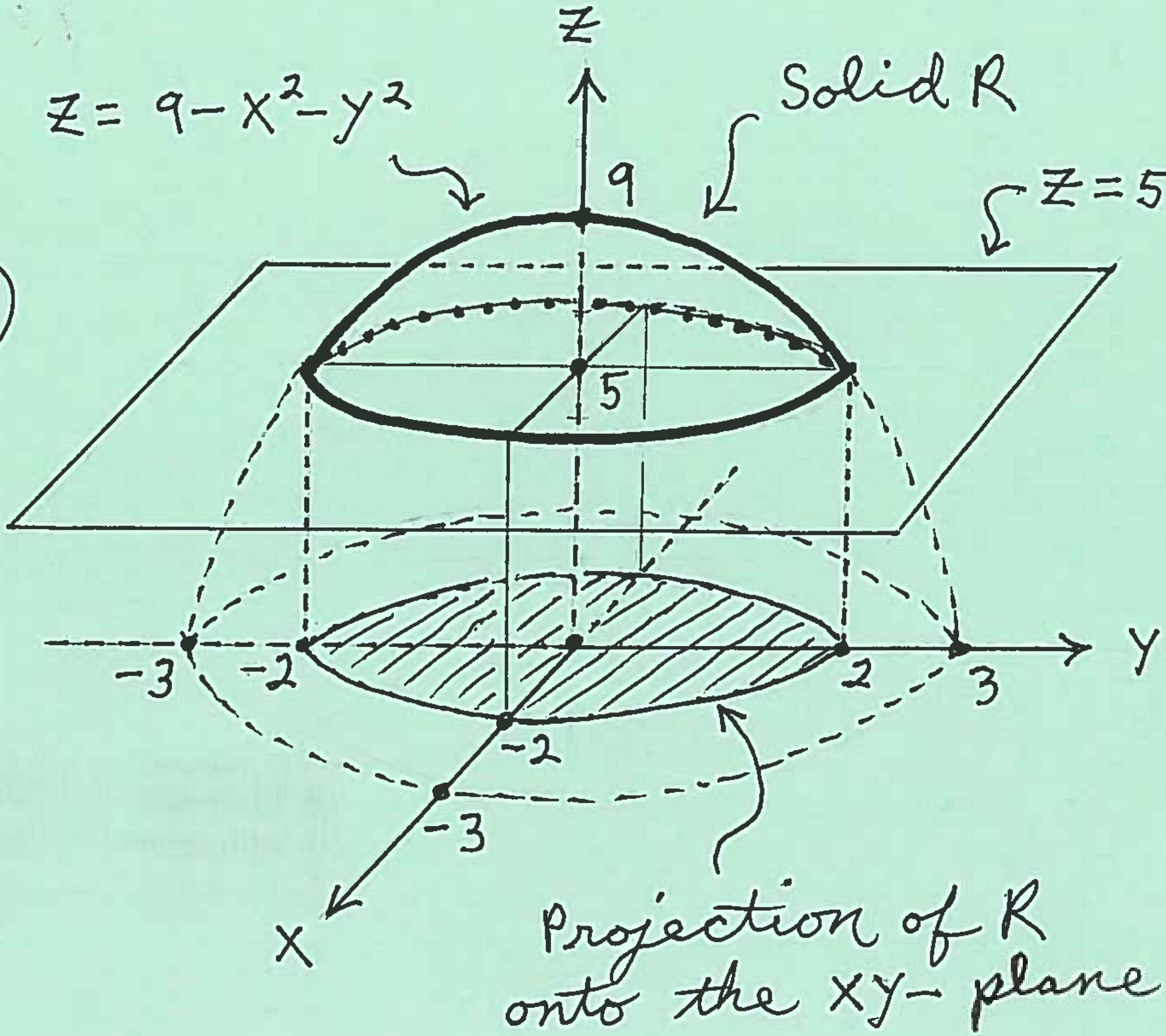
9.) Moment of Inertia of R about

a.) the origin, $(0,0,0)$

b.) the y-axis.

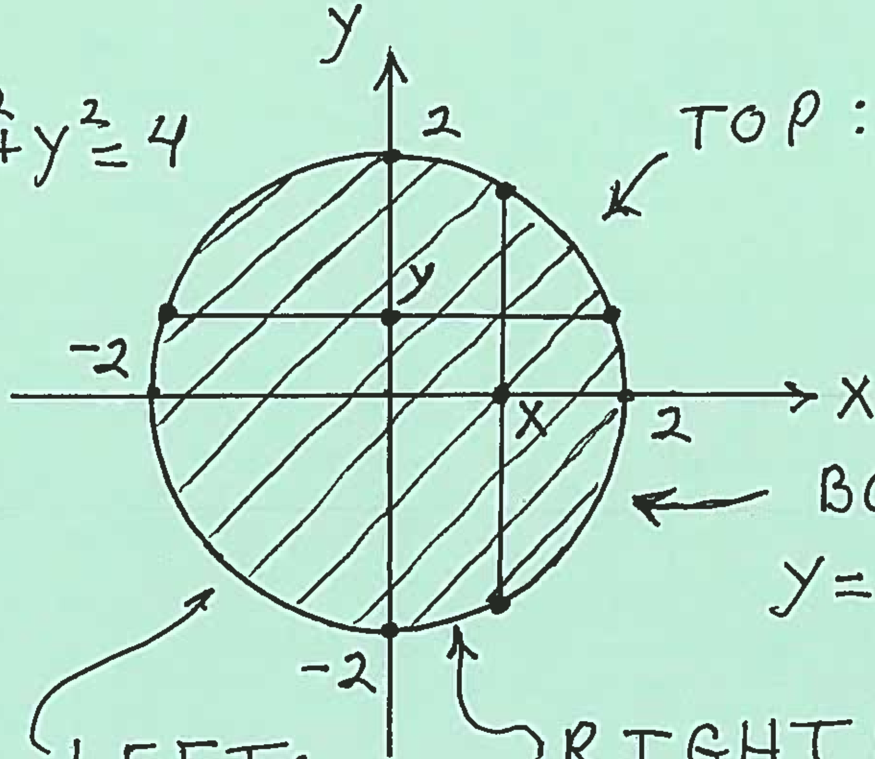
c.) line L passing through point $(3,4,5)$ and parallel to the z-axis.

1.)



2.) $Z = 9 - x^2 - y^2$ and $Z = 5 \rightarrow$
$9 - x^2 - y^2 = 5 \rightarrow x^2 + y^2 = 4 = 2^2$,
a circle of radius $r = 2$
and center $(0,0)$.

$x^2+y^2 \le 4$



TOP: $y=\sqrt{4-x^2}$

BOTTOM: $y=-\sqrt{4-x^2}$

LEFT: $x=-\sqrt{4-y^2}$

RIGHT: $x=\sqrt{4-y^2}$

a.) $\begin{cases} -2 \le x \le 2 \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \end{cases}$

b.) $\begin{cases} -2 \le y \le 2 \\ -\sqrt{4-y^2} \le x \le \sqrt{4-y^2} \end{cases}$

3.) $$\text{AREA} = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 1 \, dy \, dx \quad (m^2)$$

4.) $$\text{VOL} = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[\underbrace{(9-x^2-y^2)}_{\text{TOP}} - \underbrace{5}_{\text{BOTTOM}} \right] dz \, dy \, dx \quad (m^3)$$

OR

$$\text{VOL} = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{5}^{9-x^2-y^2} 1 \, dz \, dy \, dx \quad (m^3)$$

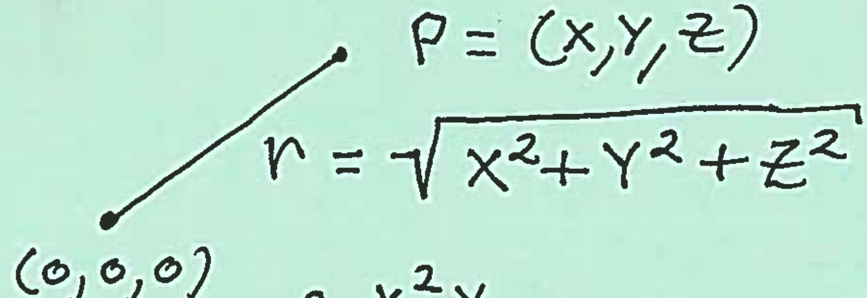
5.) $$\text{MASS} = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{5}^{9-x^2-y^2} (x+y^2+3z) \, dz \, dy \, dx \quad (kg)$$

6.) $$M_{x=4} = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{5}^{9-x^2-y^2} (x-4)(x+y^2+3z) \, dz \, dy \, dx \quad (kg \cdot m)$$

7.) $$\bar{x} = \frac{\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{5}^{9-x^2-y^2} x(x+y^2+3z)\,dz\,dy\,dx}{\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{5}^{9-x^2-y^2} (x+y^2+3z)\,dz\,dy\,dx}$$

8.) $$\bar{z} = \frac{\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{5}^{9-x^2-y^2} z\,dz\,dy\,dx}{\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{5}^{9-x^2-y^2} 1\,dz\,dy\,dx}$$

9.) a.)

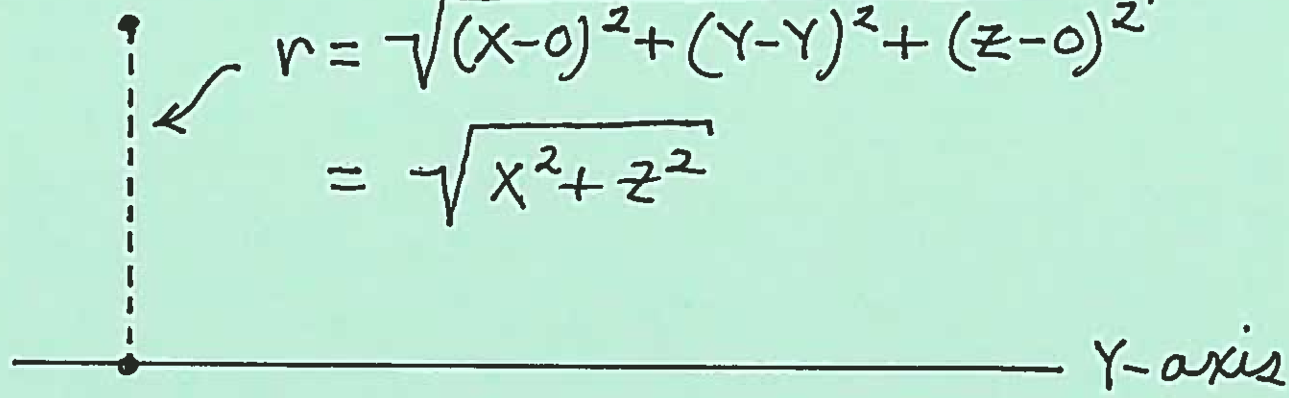


$$\text{M. of I.} = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{5}^{9-x^2-y} \left(\sqrt{x^2+y^2+z^2}\right)^2 (x+y^2+3z)\,dz\,dy\,dx$$

$(kg \cdot m^2)$

$P = (X, Y, Z)$

b.)

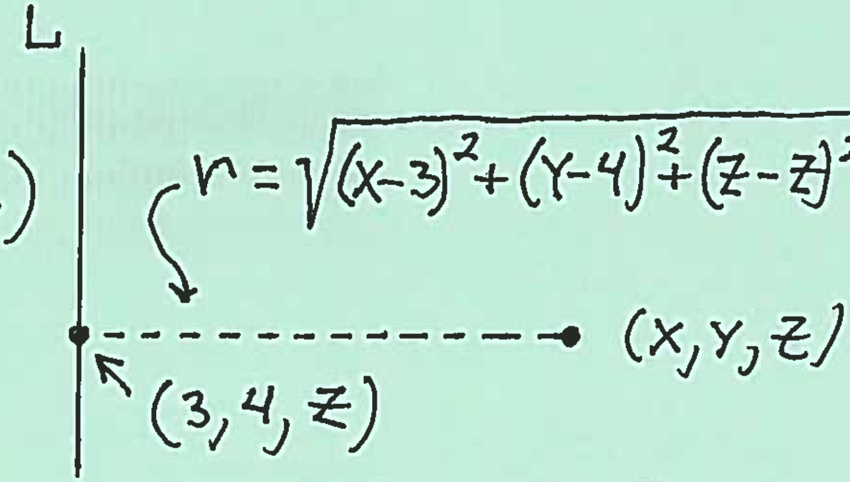


$$r = \sqrt{(X-0)^2 + (Y-Y)^2 + (Z-0)^2}$$
$$= \sqrt{X^2 + Z^2}$$

$$\text{M. of I.} = \int_{-2}^{2} \int_{-\sqrt{4-X^2}}^{\sqrt{4-X^2}} \int_{5}^{9-X^2-Y^2} \left(\sqrt{X^2+Z^2}\right)^2 (X+Y^2+3Z)\, dZ\, dY\, dX$$

$(\text{kg} \cdot \text{m}^2)$

c.)



$$r = \sqrt{(x-3)^2 + (y-4)^2 + (z-z)^2} = \sqrt{(x-3)^2 + (y-4)^2}$$

$$\text{M. of I.} = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{5}^{9-x^2-y^2} \left(\sqrt{(x-3)^2+(y-4)^2}\right)^2 (x+y^2+3z)$$
$$dz\, dy\, dx$$

$(\text{kg} \cdot \text{m}^2)$

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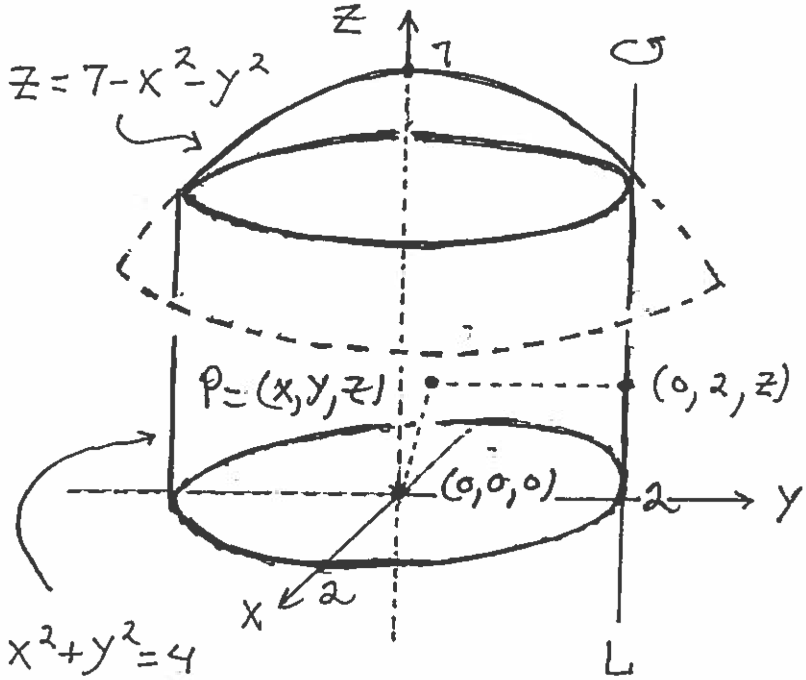
another EXAMPLE follows.

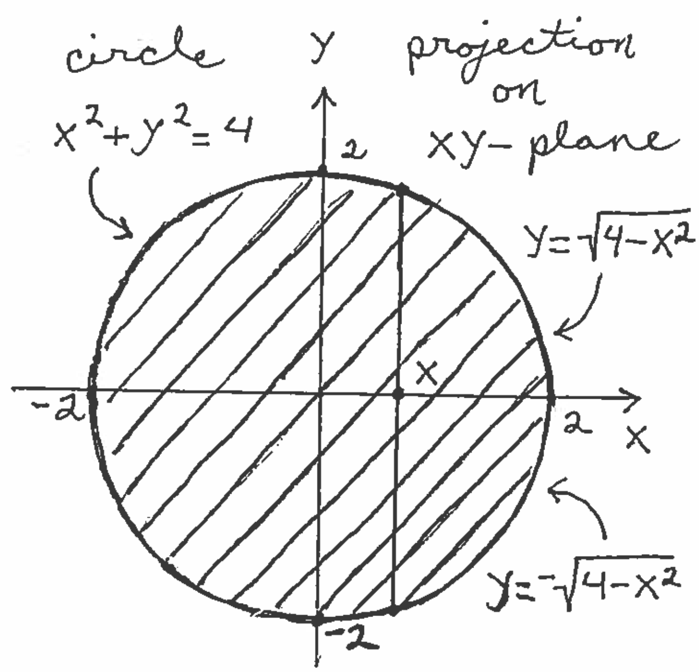
Math 21D

Kouba

Triple Integral in Rectangular Coordinates, an Example

Consider the solid region R above the plane $z=0$, inside the cylinder $x^2+y^2=4$, and below the paraboloid $z=7-x^2-y^2$. Assume that the density at point $P=(x,y,z)$ is numerically equal to the distance from P to the origin. SET UP but do not evaluate a triple integral in rectangular coordinates, which represents the *moment of inertia* of R about a line parallel to the z-axis and passing through the edge of the solid.





$$R: \begin{cases} -2 \le x \le 2 \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \\ 0 \le z \le 7-x^2-y^2 \end{cases};$$

density at P is $\boxed{\delta(x,y,z)=\sqrt{x^2+y^2+z^2}}$;

distance from P to line L is

$$\boxed{\text{distance}} = \sqrt{(x-0)^2+(y-2)^2+(z-z)^2}$$

$$\boxed{= \sqrt{x^2+(y-2)^2}} \text{ ; then}$$

$$\text{M. of I.} = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{7-x^2-y^2} \left(\sqrt{x^2+(y-2)^2}\right)^2 \cdot \sqrt{x^2+y^2+z^2}\; dz\, dy\, dx$$