Math 22A
Kouba
Spanning Sets

<u>Def</u>: Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors in a vector space V. The <u>span of S</u> is the set

$$\text{span}(S) = \{k_1\vec{v}_1 + \cdots + k_n\vec{v}_n \mid k_1, \ldots, k_n \in \mathbb{R}\}$$

<u>Ex</u>: The span of $S = \{\overrightarrow{(2,1,3)}\}$ is a line in $\mathbb{R}^3$.

<u>Ex</u>: The span of $S = \{\overrightarrow{(1,0,-1)}, \overrightarrow{(2,1,0)}\}$ is a plane in $\mathbb{R}^3$.

<u>Ex</u>: The span of $S = \{\overrightarrow{(1,0,0)}, \overrightarrow{(1,1,0)}, \overrightarrow{(1,1,1)}\}$ is all of $\mathbb{R}^3$.

Theorem: The span of $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ in a vector space V is a subspace of V.

proof: (For $r=2$) (You)

Ex: Do the following sets of vectors span $\mathbb{R}^3$?

1.) $\overrightarrow{(1,-2,3)}, \overrightarrow{(2,1,0)}, \overrightarrow{(6,-2,6)}$

2.) $\overrightarrow{(2,2,3)}, \overrightarrow{(-1,1,-2)}, \overrightarrow{(3,1,0)}$

3.) $\overrightarrow{(2,0,-1)}, \overrightarrow{(1,-1,0)}$

4.) $\overrightarrow{(1,0,-1)}$

5.) $\overrightarrow{(1,1,1)}, \overrightarrow{(2,-1,3)}, \overrightarrow{(3,0,4)}, \overrightarrow{(1,-2,2)}$

6.) $\overrightarrow{(1,0,-1)}, \overrightarrow{(1,1,0)}, \overrightarrow{(0,1,-1)},$
$\overrightarrow{(1,1,-1)}$

7.) $\overrightarrow{(1,1,0)}, \overrightarrow{(0,1,-1)}, \overrightarrow{(2,4,-2)}$

<u>SOLUTION</u>: Is

$k_1\overrightarrow{(1,1,0)} + k_2\overrightarrow{(0,1,-1)} + k_3\overrightarrow{(2,4,-2)} = \overrightarrow{(a,b,c)}$
solvable for all $a,b,c \in \mathbb{R}$? $\Rightarrow$

$$\begin{cases} k_1 + 2k_3 = a \\ k_1 + k_2 + 4k_3 = b \\ -k_2 - 2k_3 = c \end{cases} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\Rightarrow A\vec{x} = \vec{d}$ is solvable for all $\vec{d} \in \mathbb{R}^3$ iff A is invertible iff $\det(A) \neq 0$; then $\det(A)$

$$= \begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \\ 0 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0,$$ so

vectors do NOT span $\mathbb{R}^3$

8.) $\overrightarrow{(1,-1,2)}, \overrightarrow{(2,1,-1)}, \overrightarrow{(-3,2,0)}$

SOLUTION : (RECALL : $\det(A) = \det(A^T)$.)

$$\det(A) = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ -3 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 3 & -5 \\ 0 & -1 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 0 & 13 \\ 0 & -1 & 6 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & -1 & 2 \\ 0 & -1 & 6 \\ 0 & 0 & 13 \end{vmatrix} = -(1)(-1)(13) = 13 \neq 0,$$

so vectors DO span $\mathbb{R}^3$

9.) $\overrightarrow{(1,2,-2)}, \overrightarrow{(3,-1,1)}$

SOLUTION : Is

$k_1 \overrightarrow{(1,2,-2)} + k_2 \overrightarrow{(3,-1,1)} = \overrightarrow{(a,b,c)}$

solvable for all $a,b,c \in \mathbb{R}$? $\Rightarrow$

$$\begin{cases} k_1 + 3k_2 = a \\ 2k_1 - k_2 = b \\ -2k_1 + k_2 = c \end{cases} \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & a \\ 2 & -1 & b \\ -2 & 1 & c \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 3 & a \\ 0 & -7 & b-2a \\ 0 & 0 & b+c \end{array}\right] \text{ is solvable}$$

iff $b+c=0$; if $b+c \neq 0$, then the system is not solvable; so these vectors do NOT span $\mathbb{R}^3$.