The Important State Coordinates
of a Nonlinear System

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Summary. We offer an alternative way of evalating the relative importance of the state coordinates of a nonlinear control system. Our approach is based on making changes of state coordinates to bring the controllability and observability functions into input normal form. These changes of coordinates are done degree by degree and the resulting normal form is unique through terms of degree seven.

Key words: Nonlinear Control Systems, Model Reduction

1 The Problem

The theory of model reduction for linear control systems was initiated by B. C. Moore [6]. His method is applicable to controllable, observable and exponentially stable linear systems. The reduction is accomplished by making a linear change of state coordinates to simultaneously diagonalize the controllability and observability gramians and make them equal. The diagonal entries of the gramians are the singular values of the Hankel map from past inputs to future outputs. The reduction is accomplished by Galerkin projection onto the states associated to large singular values. The method is intrinsic, the reduced order model depends only on the dimension of the reduced state space.

Jonckheere and Silverman [4] extended Moore's methodology to controllable, observable but not necessarily stable linear system. Their method is based on simultaneously diagonalizing the positive definite solutions of the control and filtering Riccati equations and making them equal. The diagonal entries are called the characteristic values of the system and reduction is achieved by Galerkin projection ontothe states associated to large characteristic values. The method is sometimes called LQG balancing and reduction. Two nice features of their approach is that it is applicable to unstable systems and LQG controller of the reduced order model is the Galerkin projec-

tion of the LQG controller of the high order model. This method is intrinsic. Mustafa and Glover [7] extended Jonckheere and Silverman using H^∞ rather than LQG methods. This method is intrinsic once the attenuation level, γ has been specified.

Moore's method was extended to asymptotically stable nonlinear systems by Scherpen [8]. Scherpen and Van der Schaft [10] extended Jonckheere and Silverman to nonlinear systems and Scherpen [9] extended Mustafa and Glover. Unfortunately none of the nonlinear extensions are intrinsic, the reduced order model depends on choices made during the reduction process.

We build on the fundamental method of Scherpen and offer an alternative way of computing the reduced order model. Because of space limitations we shall restrict our attention to Moore's method and Scherpen's nonlinear generalization,

2 Linear Balancing and Reduction

2.1 Minimal Realizations

Consider an autonomous finite dimensional linear system

$$\begin{aligned} \dot{x} &= Fx + Gu \\ y &= Hx \end{aligned} \tag{1}$$

where $x \in I\!\!R^n$, $u \in I\!\!R^m$, $y \in I\!\!R^p$. The linear system (1) initialized at $x(-\infty) = 0$ defines a mapping from past inputs $\{u(s) : -\infty < s \leq 0\}$ to future outputs $\{y(t) : 0 \leq t < \infty\}$ called the Hankel map. The map factors through the state $x(0)$ at time $t = 0$. This map has infinite dimensional domain and infinite dimensional range but it factors through the finite dimensional state space $x(0) \in I\!\!R^n$ although n might be very large. The state space representation is a very succint way of describing an infinite dimensional mapping. One goal of model reduction is to reduce the state dimension as much as possible while keeping the essential features of the Hankel map.

The first step in linear model reduction is to check whether the system (1) is a minimal realization of the Hankel mapping and if it is not minimal then to reduce it to a minimal realization. This procedure is classical and goes back to Kalman and others circa 1960. We check whether the system is controllable, i.e., the system can be excited to any state $x(0)$ when started at $x(-\infty) = 0$ by using an appropriate control trajectory $\{u(s) : -\infty < s \leq 0\}$. This will be possible iff F, G is a controllable pair, i.e., the smallest F-invariant subspace $\mathcal{V}_c$ containing the columns of G is the whole state space. If the system cannot be excited to every state then we should restrict the state space to $\mathcal{V}_c$. The restricted system is controllable and has the same Hankel map.

Then we check whether this reduced system is observable in the sense that any changes in the initial condition $x(0)$ can be detected by changes in the resulting output tajectory. The system is observable iff H, F is an observable

pair, i.e., the largest F-invariant subspace $\mathcal{V}_u$ contained in the kernel of H is zero. If the system is not observable then $x(0)$ can be perturbed in the directions of $\mathcal{V}_u$ without changing the output trajectory. To make the system observable we must project $\mathcal{V}_u$ to zero. The projecteded system is observable and has the same Hankel map.

In summary, a linear system (1) is a minimal realization (of smallest state dimension) of the Hankel map iff it is controllable and observable. Any realization can be made minimal by restricting to its controllable directions and projecting out its unobservable directions.

2.2 Linear Input-Output Balancing

So far we have discussed linear systems that exactly realize the Hankel map. B. C. Moore [6] considered reduced order systems that approximately realize the Hankel map. His basic intuition was that we should ignore directions that are difficult to reach and that don't affect the output much.

To quantify these ideas, he introduced the controllablity and observability functions of the system. The controllability function is

$$\pi_c(x^0) = \inf \frac{1}{2} \int_{-\infty}^{0} |u(s)|^2 \ ds \tag{2}$$

subject to the system dynamics (1) and

$$x(-\infty) = 0, \quad x(0) = x^0.$$

If $\pi_c(x^0)$ is large then it takes a lot of input energy to excite the system in the direction x^0 and so this direction might be ignored in a reduced order model.

The observability function is

$$\pi_o(x^0) = \frac{1}{2} \int_{0}^{\infty} |y(t)|^2 \ dt \tag{3}$$

subject to the system dynamics (1) and

$$x(0) = x^0, \quad u(t) = 0.$$

If $\pi_o(x^0)$ is small then changes in this direction lead to small changes in the output energy and so this direction might be ignored in a reduced order model.

If F is Hurwitz, F, G is a controllable pair and H, F is an observable pair then it is not hard to see that

$$\pi_c(x) = \frac{1}{2} x' P_c^{-1} x, \quad \pi_o(x) = \frac{1}{2} x' P_o x$$

for some positive definite matrices P_c, P_o that are the unique solutions of the linear Lyapunov equations,

$$0 = FP_c + P_cF' + GG'$$
$$0 = F'P_o + P_oF + H'H.$$

P_c, P_o are called the controllablity and observability gramians of the system.

Moore realized that large and small are relative terms and one needs scales to measure such things. This can be accomplished by using one gramian to scale the other and vice versa. Trivally there is a linear change of state coordinates so that, in the new coordinates also denoted by x,

$$P_c = P_o = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$. These are called the Hankel singular values and they are the nonzero singular values of the Hankel map.

A reduced model can be obtained by only keeping the states corresponding to large σ_i. More pecisely suppose $\sigma_k >> \sigma_{k+1}$, let x_1 denote the first k coordinates of x and x_2 denote the remaining $n - k$ coordinates. We partition the system matrices accordingly

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u$$
$$y = \begin{bmatrix} H_1 & H_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(4)

The reduced model is then obtained by Galerkin projection onto the x_1 subspace,

$$\dot{x}_1 = F_{11}x_1 + G_1u$$
$$y = H_1x_1.$$

(5)

Notice several things. Viewed abstractly model reduction of a linear system involves injection and a surjection that is similar to minimal realiztion theory. The major difference is that in the former we need a sense of scale on $I\!R^n$ that is supplied to one gramian by the other. In minimal realization theory we did not need a scale because a direction is either controllable or not, a direction is either unobservable or not.

The eigenvalues of F play an indirect role in the reduction process. By assumption they are all in the open left half plane. It is very hard to excite the system in a direction corresponding to a very stable eigenvalue and so π_c tends to be very large in such a direction. Moreover, a state direction corresponding to a very stable eigenvalue tends to damp out quickly and so it has very little output energy as measured by π_o. Hence the very stable directions of F tend to correspond to small Hankel singular values and they tend to drop out of the reduced model.

3 Nonlinear Balancing and Reduction

Scherpen [8] generalized Moore to affine nonlinear systems of form

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x). \tag{6}$$

where the unforced dynamics $u = 0$ is asymptotically stable. She defined the controllability and observability functions (2, 3) as did Moore subject to the nonlinear system (6).

She noted that if π_c is smooth then it satisfies the Hamilton-Jacobi-Bellman equation

$$0 = \frac{\partial \pi_c}{\partial x}(x)f(x) + \frac{1}{2}\left(\frac{\partial \pi_c}{\partial x}(x)g(x)\right)\left(\frac{\partial \pi_c}{\partial x}(x)g(x)\right)' \tag{7}$$

and if it π_o smooth then it satisfies the Lyapunov equation

$$0 = \frac{\partial \pi_o}{\partial x}(x)f(x) + \frac{1}{2}|h(x)|^2. \tag{8}$$

Suppose that the system has a Taylor series expansion

$$\dot{x} = f(x) + g(x)u = Fx + Gu + O(x, u)^2$$
$$y = h(x) = Hx + O(x)^2. \tag{9}$$

If F is Hurwitz, F, G is a controllable pair, and H, F is a observable pair then it is not hard to prove that there exists locally smooth, positive definite solutions to the above PDEs and

$$\pi_c(x) = \frac{1}{2}x'P_c^{-1}x + O(x)^3$$

$$\pi_o(x) = \frac{1}{2}x'P_ox + O(x)^3$$

where P_c, P_o are the controllability and observability gramians defined above.

So far nonlinear balancing looks very much like linear balancing but, in general, there is not a nonlinear change of state coordinates that simultaneously "diagonalizes" both $\pi_c(x)$ and $\pi_o(x)$.

So Scherpen invoked the Morse lemma to show that after a nonlinear change of state coordinates

$$\pi_c(x) = \frac{1}{2}|x|^2,$$
$$\pi_o(x) = \frac{1}{2}x'Q(x)x, \quad Q(0) = P_o.$$

Then after a further nonlinear change of coordinates $\pi_c(x)$ is unchanged and

$$\pi_o(x) = \frac{1}{2}x' \begin{bmatrix} \tau_1(x) & & 0 \\ & \ddots & \\ 0 & & \tau_n(x) \end{bmatrix} x$$

where $\tau_i(x)$ are called the singular value functions. It is not hard to see that $\tau_i(0) = \sigma_i^2$ where σ_i are the Hankel singular values of the linear part of the system.

The Hankel singular values σ_i of the linear part of the system are intrinsic and hence so are their squares, $\tau_i(0)$. But the singular value functions $\tau_i(x)$ are not [3]. For example, choose any two distinct indices $i \neq j$ and any $c \in I\!\!R$. Define $\bar{\tau}_i(x) = \tau_i(x) + cx_j^2$, $\bar{\tau}_j(x) = \tau_j(x) - cx_i^2$ and $\bar{\tau}_k(x) = \tau_k(x)$ otherwise. Then $\pi_c(x)$ is unchanged and

$$\pi_o(x) = \frac{1}{2}x' \begin{bmatrix} \bar{\tau}_1(x) & & 0 \\ & \ddots & \\ 0 & & \bar{\tau}_n(x) \end{bmatrix} x$$

Scherpen's next step was to make an additional change of coordinates so that if x is in a coordinate direction $x = (0, \ldots, x_i, \ldots, 0)$ then

$$\pi_c(x) = \frac{1}{2}\bar{\sigma}_i(x_i)^{-1}x_i^2$$
$$\pi_o(x) = \frac{1}{2}\bar{\sigma}_i(x_i)x_i^2$$

where $\bar{\sigma}_i(0) = \sigma_i$ and $\bar{\sigma}_i(x_i)^2 \approx \tau_i(x)$.

Scherpen obtain a reduced order model by neglecting states with small $\bar{\sigma}_i(x)$. Suppose for all $x \in \mathcal{X}$, a neighborhood of $0 \in I\!\!R^n$,

$$\bar{\sigma}_1(x) \geq \ldots \geq \bar{\sigma}_k(x) >> \bar{\sigma}_{k+1}(x) \geq \ldots \geq \bar{\sigma}_n(x) > 0$$

then as before we partition $x = (x_1, x_2)$ and Galerkin project onto the states cooresponding to large $\bar{\sigma}_i(0)$.

Unfortuantely this approach to obtaining a reduced order model is not intrinsic. The resulting reduced order system is not independent of the particular coordinate changes that led to it. Also it depends on the choice of singular value functions $\tau_i(x)$.

One nice feature of this approach is that the controllability function of the reduced order model is the restriction of the controllability function of the full order model. However this is not true for the observability functions but they do agree to $O(x)^3$.

4 The New Approach

Following Moore and Scherpen we consider the optimal control problem of steering from $x = 0$ at $t = -\infty$ to an arbitrary x at $t = 0$ while minimizing the energy of the input

$$\pi_c(x) = \inf \frac{1}{2} \int_{-\infty}^{0} |u|^2 dt$$

for the system

$$\begin{aligned} \dot{x} &= f(x, u) = Fx + Gu + f^{[2]}(x, u) + \ldots \\ y &= h(x) = Hx + h^{[2]}(x) + \ldots. \end{aligned} \tag{10}$$

where $f^{[d]}(x, u), h^{[2]}(x)$ denotes homogeneous polynomials of degree d. Scherpen only considered systems affine in u but it is an easy generalization to the above.

If F is Hurwitz and F, G is a controllable pair then there is an unique, locally smooth and positive definite optimal cost $\pi_c(x)$ and an unique, locally smooth optimal control $u = \kappa(x)$ which solve the HJB equations

$$0 = \frac{\partial \pi_c}{\partial x}(x) f(x, \kappa(x)) - \frac{1}{2} |\kappa(x)|^2 \tag{11}$$

$$\kappa(x) = \left(\frac{\partial \pi_c}{\partial x}(x) \frac{\partial f}{\partial u}(x, \kappa(x)) \right)' \tag{12}$$

Moreover, following Al'brecht [1], the Taylor series of $\pi_c(x), \kappa(x)$ can be computed term by term from the Taylor series of $f(x, u)$,

$$\pi_c(x) = \frac{1}{2} x' P_c^{-1} x + \pi_c^{[3]}(x) + \ldots + \pi_c^{[r]}(x) + O(x)^{r+1}$$

$$\kappa(x) = Kx + \kappa^{[2]}(x) + \ldots + \kappa^{[r-1]}(x) + O(x)^r$$

where $P_c > 0$ and $K = G'P_c^{-1}$ are the controllability gramian and the optimal feedback of the linear part of the system.

As before we also consider the output energy released by the system when it starts at an arbitrary x at $t = 0$ and decays to 0 as $t \to \infty$,

$$\pi_o(x) = \frac{1}{2} \int_0^\infty |y|^2 dt.$$

If F is Hurwitz and H, F is an observable pair then there is a unique locally smooth and positive definite solution $\pi_o(x)$ to the corresponding Lyapunov equation

$$0 = \frac{\partial \pi_o}{\partial x}(x) f(x) + \frac{1}{2} h'(x) h(x) \tag{13}$$

Again the Taylor series of $\pi_o(x)$ can be computed term by term from the Taylor series of f and h,

$$\pi_o(x) = \frac{1}{2} x' P_o x + \pi_o^{[3]}(x) + \ldots + \pi_o^{[r]}(x) + O(x)^{r+1}$$

where $P_o > 0$ is the observability gramianof the linear part of the system.

From [6], [8] we know that we can choose a linear change of coordinates so that in the new coordinates also denoted by x

$$\pi_c(x) = \frac{1}{2} |x|^2 + \pi_c^{[3]}(x) + O(x)^4$$

$$\pi_o(x) = \frac{1}{2} x' \begin{bmatrix} \tau_1 & & 0 \\ & \ddots & \\ 0 & & \tau_n \end{bmatrix} x + \pi_o^{[3]}(x) + O(x)^4$$

where the so called singular values $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_n > 0$ are the ordered eigenvalues of $P_o P_c$. If this holds then we say that the system is in *input normal form of degree one.* Because of space limitations we shall restrict our attention to the generic case where the singular values $\tau_1 > \tau_2 > \ldots > \tau_n > 0$ are distinct.

A system with distinct singular values is in *input normal form of degree d* if

$$\pi_c(x) = \frac{1}{2} \sum_{i=1}^{n} x_i^2 + O(x)^{d+2}$$

$$\tag{14}$$

$$\pi_o(x) = \frac{1}{2} \sum_{i=1}^{n} \eta_i^{[0:d-1]}(x_i) x_i^2 + O(x)^{d+2}$$

where $\eta_i^{[0:d-1]}(x_i) = \tau_i + \ldots$ is a polynomial in x_i with terms of degrees 0 through $d - 1$. They are called the squared singular value polynomials of degree $d - 1$.

There is also an *output normal form of degree d* where the forms of $\pi_c(x)$ and $\pi_o(x)$ are reversed.

The proof of the following is omitted because of page limitations. The full details can be found in [5].

Theorem Suppose the system (10) is C^r, $r \geq 2$ with controllable, observable and exponentially stable linear part. If the τ_i are distinct and if $d < r - 1$ then there is a change of state coordinates that takes the system into input normal form of degree d (14). The change of coordinates that achieves the input normal form of degree d is not necessarily unique but the input normal form of degree $d \leq 6$ is unique. If f, h are odd functions the input normal form of degree $d \leq 12$ is unique.

The differences between input normal form of degree d and Scherpen's normal form are threefold. First the former is only approximate through terms of degree $d + 1$ while the latter is exact. The second difference is that in

the former the parameters $\eta_i^{[0:d-1]}(x_i)$ only depend on x_i while in the latter the parameters $\tau_i(x)$ can depend on all the components of of x. Thirdly the parameters $\eta_i^{[0:d-1]}(x_i)$ of the former are unique if $d \leq 6$ while the parameters $\tau_i(x)$ of the latter are not unique except at $x = 0$ [3] .

Recently Fujimoto and Scherpen [2] have shown the existence of a normal form where π_c is one half the sum of squares of the state coordinates and

$$\frac{\partial \pi_o}{\partial x_i}(x) = 0 \ \text{ iff } \ x_i = 0. \tag{15}$$

It is closer to our input normal form of degree d (14) which has similar properties. The controllability function π_c is one half the sum of squares of the state coordinates through terms of degree $d + 1$ and

$$\frac{\partial \pi_o}{\partial x_i}(x) = O(x)^{d+1}$$

if $x_i = 0$. But the normal form of Fujimoto and Scherpen is not unique while the input normal form of degree $d \leq 6$ is unique.

Notice that if a system with distinct singular values $\tau_i = \tau_i(0)$ is in input normal form of degree d then its controllability and observability functions are "diagonalized" through terms of degree $d + 1$. They contain no cross terms where one coordinate multiplies a different coordinate. This is reminiscent of the balancing of linear systems by B. C. Moore [6].

For linear systems the singular value τ_i is a measure of the importance of the coordinate x_i. The "input energy" needed to reach the state x is $\pi_c(x)$ and the "output energy" released by system from the state x is $\pi_o(x)$. The states that are most important are those with the most "output energy" for fixed "input energy". Therefore in constructing the reduced order model, Moore kept the states with largest τ_i for they have the most "output energy" per unit "input energy".

In Scherpen's generalization [8] of Moore, the singular value functions $\tau_i(x)$ measure the importance of the state x_i. To obtain a reduced order model, she assumed $\tau_i(x) > \tau_j(x)$ whenever $1 \leq i \leq k < j \leq n$ and x is in a neighborhood of the origin. Then she kept the states $x_1, \ldots, x_k$ in the reduced order model. But the $\tau_i(x)$ are not unique.

For nonlinear systems in input normal form of degree d, the polynomial $\eta_i^{[0:d-1]}(x_i)$ is a measure of the importance of the coordinate x_i for moderate sized x. If the τ_i are distinct and $d \leq 6$ then $\eta_i^{[0:d-1]}(x_i)$ is unique. The leading coefficient of this polynomial is the singular value τ_i so in constructing a reduced order model we will want to keep the states with the largest τ_i. But τ_i can be small yet $\eta_i^{[0:d-1]}(x_i)$ can be large for moderate sized x_i. If we are interested in capturing the behavior of the system for moderate sized inputs, we may also want to keep such states in the reduced order model.

To obtain a reduced order model we proceed as follows. We start by making a linear change of coordinates to take the system into input normal form

of degree 1. If the singular values are distinct this change of coordinates is uniquely determined up to the signs of the coordinates. In other words replacing x_i by $-x_i$ does not change the input normal form of degree 1. Next one computes the Taylor series expansions to degree $d+1$ of the controllability and observability functions, $\pi_c(x)$, $\pi_o(x)$. Then degree by degree one makes changes of state coordinates to bring the system into input normal form of degree d. The input normal form of degree d are intrinsic but the changes of state coordinates that achieve are not. We defer for a later paper [5] the discussion of which changes should be used . Suppose that the input energies that we shall use are all less than $\frac{c^2}{2}$ for some constant $c > 0$. Then we expect the system to operate in $|x| < c$ where x are the input normal coordinates of degree d. We compare the sizes of $\eta_i^{[0:d-1]}(x_i)$ for $|x_i| < c$ and split them into two categories, large and small. The reduced order model is obtained by Galerkin projection onto the coordinates corresponding to the large $\eta_i^{[0:d-1]}(x_i)$.

5 Conclusion

We have developed a way of finding state coordinates that lend themselves to measuring there relative importance. The measure of importance is unique up to degree 6 (degree 12 for odd systems). Unfortunately the coordinates are not unique beyond degree one. Since a reduced order model is obtained by Galerkin projection in these coordinates, it is not unique. Further research is needed to clarify these issues.

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