7.1.8) a) $\|F-f\|^2 = \int_{-\infty}^{\infty} |F(x)-f(x)|^2\,dx$

$$= \int_{-\infty}^{-N} |f(x)|^2 dx + \int_{-N}^{N} |F(x)-f(x)|^2 dx + \int_{N}^{\infty} |f(x)|^2 dx$$

$$= \int_{-\infty}^{-N} |f(x)|^2 dx + \int_{N}^{\infty} |f(x)|^2 dx.$$

So $\lim_{N\to\infty} \|F-f\|^2 = 0$, hence N can be chosen large enough that $\|F-f\| < \frac{1}{2}\delta$.

b) g is C^∞ by Thm 7.2 (applied to K); vanishes outside a finite interval by definition of F.
To show that $\|f-g\| < \delta$, note that

$$\|f-g\| = \|f - F*K_\varepsilon + F*K_\varepsilon - g\|$$

$$\leq \|f - F*K_\varepsilon\| + \|F*K_\varepsilon - g\|,$$

the result follows by taking $\varepsilon \to 0$.

7.2.1) $\int_{-\infty}^{\infty} e^{-i\xi x - ax^2/2}\, dx = \int_{-\infty}^{\infty} e^{-\frac{a}{2}(x+\frac{i\xi}{a})^2} e^{-\frac{a}{2}\cdot\frac{\xi^2}{a^2}} dx$

$= e^{-\xi^2/2a} \int e^{-\frac{a}{2}(x+\frac{i\xi}{a})}\, dx$

$= e^{-\xi^2/2a} \int_{-\infty}^{\infty} e^{-\frac{a}{2}y^2}\, dy$

$= \sqrt{\frac{2\pi}{a}}\, e^{-\xi^2/2a}$

7.2.2) $\int_{-\infty}^{\infty} e^{-i\xi x} e^{-a|x|} dx = \int_{-\infty}^{0} e^{-i\xi x} e^{ax}\, dx + \int_{0}^{\infty} e^{-i\xi x} e^{-ax}\, dx$

$= \int_{-\infty}^{0} e^{x(a-i\xi)}\, dx + \int_{0}^{\infty} e^{x(-i\xi-a)}\, dx$

$= \frac{1}{a-i\xi} + \frac{1}{a+i\xi} = \frac{2a}{a^2+\xi^2}$

Now, $e^{-a|x|} = \frac{1}{2\pi}\int e^{i\xi x}\cdot 2a(a^2+\xi^2)^{-1}\, d\xi$

7.2.3) a) $F(x-a) = \int e^{-i\xi x} f(x-a)\, dx$,

set $y = x-a$, $= \int e^{-i\xi(y+a)} f(y)\, dy$

$= e^{-i\xi} \int e^{-i\xi y} f(y)\, dy$

$= e^{-i\xi a} \hat{f}(\xi)$

$F(e^{iax} f(x)) = \int e^{i\xi x} e^{iax} f(x)\, dx$

$= \int e^{-ix(\xi-a)} f(x)\, dx$

set $z = \xi - a$,

$= \int e^{-izx} f(x)\, dx$

$= f(z) = \hat{f}(\xi - a)$

b) $f_\delta(x) = \delta^{-1} f\left(\frac{x}{\delta}\right)$

$\hat{f}_\delta = \int e^{-ix\xi} f_\delta(x)\, dx$

$= \int e^{-ix\xi} \delta^{-1} f\left(\frac{x}{\delta}\right) dx$ set $y = \frac{x}{\delta}$,

$= \int e^{-i\xi\delta y} f(y)\, dy$

$= \hat{f}(\delta\xi)$

7.2.5 (a) $\hat{g}(\delta\xi) = \int g(x) e^{-i\delta\xi x} dx$ is well-defined for all δ and ξ, since $g \in L^1$. Moreover, since $|g(x) e^{-i\delta\xi x}| \le |g(x)|$, for all x, we apply Dominated Convergence Thm (pg 83) to interchange the order of the limit and the integration,

$$\lim_{\delta\to 0} \int g(x) e^{-i\delta\xi x} dx = \int g(x) \left(\lim_{\delta\to 0} e^{-i\delta\xi x} \right) dx$$

$$= \int g(x) dx = 1 .$$

(b) Suppose f is P.C. and $f \in L^1$. Let $\alpha = \int_{-\infty}^{0} g(x) dx$, $\beta = \int_0^\infty g(x) dx$. Note $\alpha + \beta = 1$.

Claim: $\lim_{\delta\to 0} \frac{1}{2\pi} \int e^{i\xi x} \hat{g}(\delta\xi) \hat{f}(\xi) d\xi = \lim_{\delta\to 0} (f * g_\delta)(x)$

$$= \alpha f(x^+) + \beta f(x^-), \text{ by Thm 7.3.}$$

Thus, if f is continuous, then $f(x^+) = f(x^-) = f(x)$, and the limit is $(\alpha+\beta) f(x) = f(x)$, exactly what we wanted.

The claim is easy to prove:

$$\frac{1}{2\pi} \int e^{i\xi x} \hat{g}(\delta\xi) \hat{f}(\xi) d\xi = \frac{1}{2\pi} \int e^{i\xi x} \hat{g}(\delta\xi) \left(\int f(y) e^{-i\xi y} dy \right) d\xi$$

$$= \int \frac{1}{2\pi} \int e^{i\xi(x-y)} \hat{g}(\delta\xi) \, d\xi \, f(y) \, dy$$

$$= \int \mathcal{F}^{-1}[\hat{g}(\delta\xi)](x-y) f(y) dy$$

$$= \int g_\delta(x-y) f(y) dy \quad , \text{ by Thm 7.5(b)}$$

$$= (f * g_\delta)(x) .$$

7.2.13 Plancherel Thm : $\int g(t)\overline{f(t)}\,dt = \frac{1}{2\pi}\int \hat{g}(\xi)\overline{\hat{f}(\xi)}\,d\xi$

(a) $F\left[\frac{\sin(at)}{t}\right] = \pi \chi_a(\xi)$

$$\Rightarrow \int \frac{\sin(at)\sin(bt)}{t^2}\,dt = \frac{\pi^2}{2\pi}\int \chi_a(\xi)\chi_b(\xi)\,d\xi$$

$$= \frac{\pi}{2}\int_{-\min(a,b)}^{\min(a,b)} d\xi = \pi \min(a,b).$$

(b) $F[x f(x)] = i\frac{d}{d\xi}\hat{f}(\xi)$, $F\left[\frac{1}{t^2+a^2}\right] = \frac{\pi}{a}e^{-a|\xi|}$, and

$$F\left[\frac{t}{t^2+a^2}\right] = i\frac{\pi}{a}\frac{d}{d\xi}\left[e^{-a|\xi|}\right] = i\frac{\pi}{a}\cdot(-H(\xi))\cdot a e^{-a|\xi|}$$

where $H(z) = \begin{cases} 1, & \text{if } z>0 \\ -1, & \text{if } z<0 \end{cases}$.

$$\int \frac{t^2}{(t^2+a^2)(t^2+b^2)}\,dt = \frac{1}{2\pi}\left(\frac{\pi^2}{ab}\cdot ab\int (-H(\xi))^2 e^{-a|\xi|-b|\xi|}\,d\xi\right)$$

$$= \frac{\pi}{2}\int e^{-(a+b)|\xi|}\,d\xi$$

$$= -\frac{\pi}{2}\left[\frac{e^{+(a+b)\xi}}{-(a+b)}\right]_{-\infty}^{0} - \frac{\pi}{2}\left[\frac{e^{-(a+b)\xi}}{a+b}\right]_{0}^{\infty}$$

$$= \frac{\pi}{a+b}.$$