7.5.1 (a) Take Fourier Transform, $\mathcal{F}[u_{xx}+2u_{yy}+3u_x-4u]=\hat{f}$

$\mathcal{F}[u_x] = i\xi_1\hat{u}(\xi_1,\xi_2)$ and $\mathcal{F}[u_y] = i\xi_2\hat{u}(\xi_1,\xi_2)$

$\Rightarrow$ $$-\xi_1^2\hat{u}(\xi_1,\xi_2) - 2\xi_2^2\hat{u}(\xi_1,\xi_2) + 3i\xi_1\hat{u}(\xi_1,\xi_2) - 4\hat{u}(\xi_1,\xi_2) = \hat{f}(\xi_1,\xi_2).$$

$$\Rightarrow (-\xi_1^2 - 2\xi_2^2 + 3i\xi_1 - 4)\hat{u} = \hat{f}.$$

Now, $\xi_1,\xi_2 \in \mathbb{R}$, so $-2\xi_2^2+3i\xi_1-4 \neq 0$ in the Complex plane. Therefore we can divide the eqn by this to get

$$\hat{u}(\xi_1,\xi_2) = \frac{\hat{f}(\xi_1,\xi_2)}{-\xi_1^2-2\xi_2^2+3i\xi_1-4}$$

$$\therefore \quad u(x,y) = \frac{1}{(2\pi)^2}\iint \frac{\hat{f}(\xi_1,\xi_2)}{-\xi_1^2-2\xi_2^2+3i\xi_1-4}\, d\xi_1 d\xi_2 .$$

(b) $(\xi_1^4+\xi_2^2+2)\hat{u}(\xi_1,\xi_2) = \hat{f}(\xi_1,\xi_2)$

$\xi_1^4+\xi_2^2+2 \neq 0$ for all $\xi_1,\xi_2 \in \mathbb{R}$.

$$u(x,y) = \frac{1}{(2\pi)^2}\iint \frac{\hat{f}(\xi_1,\xi_2)}{\xi_1^4+\xi_2^2+2}\, d\xi_1 d\xi_2 .$$

7.5.6

(a) $\hat{f}(\xi) = \tilde{f}_0(\rho) = 2\pi \int_0^\infty f_0(r) J_0(r\rho)\, r\, dr$

where $f_0(r) = \begin{cases} 1, & |x| = r \leq 1 \\ 0, & \text{otherwise} \end{cases}$

$\Rightarrow \hat{f}(\xi) = 2\pi \int_0^1 J_0(r\rho)\, r\, dr$, let $r' = r\rho$,

$= 2\pi \int_0^\rho J_0(r') \frac{r'}{\rho} \cdot \frac{dr'}{\rho}$

$= \frac{2\pi}{\rho^2} \int_0^\rho J_0(r')\, r'\, dr' = \frac{2\pi}{\rho^2} \left(\rho J_1(\rho)\right) = \frac{2\pi J_1(\rho)}{\rho}.$

(b) $\hat{f}(\xi) = \tilde{f}_0(\rho) = \frac{4\pi}{\rho} \int_0^\infty f_0(r)\, r \sin r\rho\, dr$

$= \frac{4\pi}{\rho} \int_0^\rho \frac{r'}{\rho} \sin r' \frac{dr'}{\rho} = \frac{4\pi}{\rho^3} \left[-r' \cos r' \Big|_0^\rho + \int_0^\rho \cos r'\, dr' \right]$

$= \frac{4\pi}{\rho^3} \left[-\rho \cos\rho + \sin\rho \right]$

AE1

$$\widehat{(a*b)}(u) = \frac{1}{N}\sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} a(k)b([n-k])\right)e^{-2\pi i un/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} a(k)\left(\frac{1}{N}\sum_{n=0}^{N-1} b[n-k]\,e^{-2\pi i un/N}\right)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} a(k)\left(\frac{1}{N}\left(\sum_{n=0}^{k-1} b(n-k+N) + \sum_{n=k}^{N-1} b(n-k)\right)e^{-2\pi i un/N}\right)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} a(k)\left[\frac{1}{N}\sum_{m=N-k}^{N-1} b(m)\,e^{-2\pi i u(m+k-N)/N} + \frac{1}{N}\sum_{m=0}^{N-k-1} b(m)\,e^{-2\pi i u(m+k)/N}\right]$$

since $e^{-2\pi i u(m+k-N)} = e^{-2\pi i u(m+k)/N}$

$$= \frac{1}{N}\sum_{k=0}^{N-1} a(k)\left(\frac{1}{N}\sum_{m=0}^{N-1} b(m)\,e^{-2\pi i u(m+k)/N}\right)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} a(k)\,e^{-2\pi i u\frac{k}{N}}\left(\frac{1}{N}\sum_{m=0}^{N-1} b(m)\,e^{-2\pi i um/N}\right)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} a(k)\,e^{-2\pi i uk/N}\,\hat{b}(u) = \hat{a}(u)\,\hat{b}(u).$$

AE2

$$F(u,v) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(-1)^{m+n} f(m,n)\,e^{-2\pi i\left(\frac{um}{M}+\frac{vn}{N}\right)}$$

$$= \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{-\pi i(m+n) - 2\pi i\left(\frac{um}{M}+\frac{vn}{N}\right)} f(m,n)$$

$$= \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{-2\pi i\left((u+\frac{M}{2})\frac{m}{M} + (v+\frac{N}{2})\frac{n}{N}\right)} f(m,n)$$

$$= \hat{f}\left(u+\frac{M}{2},\ v+\frac{N}{2}\right)$$

AE3 (a) First, for $m=n=0$,

$$h(m,n) = \frac{1}{MN}\sum_{=0}^{M-1}\sum_{=0}^{N-1} A = \frac{A}{M\cdot N}\cdot M\cdot N = A.$$

For $m \neq 0$:

$$h(m,n) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} A e^{+2\pi i\left(\frac{um}{M}+\frac{vn}{N}\right)}$$

$$= \frac{A}{MN}\underbrace{\left(\frac{1-e^{2\pi i m}}{1-e^{2\pi i m/N}}\right)}_{=0}\cdot\sum_{v=0}^{N-1} e^{2\pi i \frac{vn}{N}} = 0$$

Similarly, for $n \neq 0$.

$\therefore\ h(m,n) = A\delta(m,n)$.

(b)

$$(f * h)(m,n) = \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} f(k,l)\, h([m-k,\ n-l])$$

$$= \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} f(k,l)\, A\delta([m-k,\ n-l])$$

$$= \frac{A}{MN} f(m,n).$$

(So, the resulting effect in spatial domain is multiplication of $f(m,n)$ by the same constant $\frac{A}{MN}$!!!