COVERING SPACES

Reading: Lee, *Introduction to Topological Manifolds*, Chapters 5,6

A covering map $f : X- > Y$ is a function such that for every point $y \in Y$, there is a neighborhood V of Y such that $\pi^{-1}(V)$ is a disjoint union $U_1 \sqcup \cdots \sqcup U_d$ (d may be infinite if X is non-compact) such that each U_i is homeomorphic to V. We say f is a d-fold covering map.

(1) The first examples are coverings of the circle by itself. Consider the circle as the complex numbers z of unit length $|z| = 1$. Then define $p : S^1- > S^1$ by $p(z) = z^d$. Check this is a d-fold covering map.

(2) Similarly the real line is an infinite cover of the circle. The covering map is $f : \mathbb{R}- > S^1$ defined by $f(t) = (cos(t), sin(t))$. Check this is a covering map.

The next thing to look at is if $p : X- > Y$ is a covering map between two surfaces, what can you say about X and Y?

(3) The key theorem to try to prove is: If $p : X- > Y$ is a d-fold covering map between compact surfaces X and Y, then the Euler characteristic $\chi(X) = d\chi(Y)$. To prove this, try to use a cell decomposition of Y and the covering map p to define a cell decomposition of X where each of the cells of Y appears d times in the cell decomposition of X.

(4) Using this theorem, you can get restrictions on which surfaces can cover each other. See if you can find covering maps between surfaces when the Euler characteristic allows it and find families of surfaces where there is no covering map between them.