MAT 145: Homework 5 Solution

Xiaochen Liu*

1. (a) $\#V(K_{m,n}) = m + n$. $\#E(K_{m,n}) = mn$.

   (b) The sequence is $(m, \dots, m, n, \dots, n)$, where there are n of m's and m of n's

2. (a) $(2, 2, \dots, 2, 1, 1)$, where there are $n - 2$ of 2's.

   (b) $(n - 1, n - 1, \dots, n - 1)$, where there are n terms.

   (c) $(4, 3, 2, 2, 1, 1)$.

3. Let the map $f : V(G) \to V(K_n)$ be any bijective map. (Existence is clear, as both of the sets have cardinality n.) Let $K_n' = (V(K_n), E(K_n)')$, where $E(K_n)' = \{f(a)f(b) : ab \in E(G)\}$. Then $E(K_n)' \subset E(K_n)$ as K_n is complete and thus there is an edge connecting every pair of vertices. So, K_n' is a subgraph of K_n and K_n' is isomorphic to G.

4. (a) Impossible. There is 1 (odd number) node with odd degree.

   (b) A square with one pair of the diagonal connected. Graph omitted.

   (c) A cycle with n nodes. Graph omitted.

   (d) Impossible. There are 3 (odd number) nodes with odd degree.

5. Let f be the isomorphism between G_1 and G_2, i.e., for every $x, y \in V(G_1), xy \in E(G_1)$ if and only if $f(x)f(y) \in E(G_2)$. Let $a_i \in V(G_1)$ and $deg(a_i) = d_i$. Then it suffices to show that $deg(f(a_i)) = d_i$. Let $V_a(G_1) = \{b \in V(G_1) : ab \in E(G_1)\}$. Then $d_i = \#V_{a_i}(G_1)$, where $\#$ means cardinality of a set. Now, note that $b \in V_{a_i}(G_1)$ if and only if $a_i b \in V(G_1)$ if and only if $f(a_i)f(b) \in E(G_2)$ if and only if $f(b) \in V_{f(a_i)}(G_2)$. So, $d_i = \#V_{f(a_i)}(G_2) = deg(f(a_i))$.

6. We can easily check that there is a path connecting any two nodes. Details omitted.

7. Let f be the isomorphism from G_1 to G_2. Let $e = ab$ be a cutting edge of G_1, i.e., $G_1' = (V(G_1), E(G_1) \setminus \{e\})$ is not connected. That is, there exist two nodes in G_1', say x, y, such that there is no path connecting x and y.

*xchliu@math.ucdavis.edu

We claim that $f(a)f(b)$ is a cutting edge of G_2, i.e., $G_2' = (V(G_2), E(G_2) \setminus \{f(a)f(b)\})$ is not connected.

Otherwise, in G_2', there is a path between $f(x)$ and $f(y)$, e.g., $f(x) \to x_1 \cdots \to x_n \to f(y)$. Note that in this path, no edge is $f(a)f(b)$. Then in G_1', $f^{-1}(f(x)) \to f^{-1}(x_1) \cdots \to f^{-1}(x_n) \to f^{-1}(f(y))$, (no edge is ab) i.e., $x \to f^{-1}(x_1) \cdots \to f^{-1}(x_n) \to y$ is a path connecting x and y.

8. (a) An isomorphism is given by sending x to x', for $x \in \{a, b, c, d, e, f\}$.

(b) Not isomorphic. The left graph has more nodes than the right one.

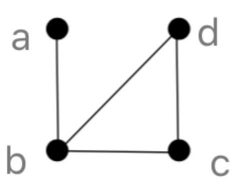
(c) Not isomorphic. The left graph has more edges than the right one.

(d) An isomorphism is given by sending x to x', for $x \in \{a, b, c, d, e, f\}$.

(e) Not isomorphic. The left graph has more edges than the right one.

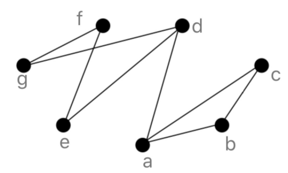
(f) Not isomorphic. The left graph has less edges than the right one.

9. (a) $a \to b \to c \to d \to b$

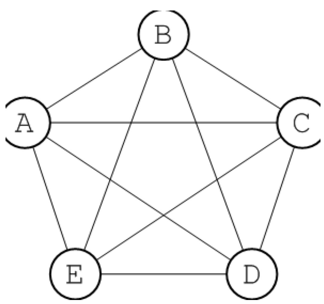


(b) No Eulerian walk, as there are 4 (>2) nodes with odd degree.

(c) $a \to b \to c \to a \to d \to e \to f \to g \to d$



(d) $A \to B \to C \to D \to E \to A \to C \to E \to B \to D$.



(e) No Eulerian walk, as there are 6 (>2) nodes with odd degree 5.