Midterm

Math 145, Spring 2019

Name: Solutions

Student ID: ____________________

Every solution must contain an explanation written in words supporting your numerical solution to receive credit.

You do not need to simplify numerical expressions for your final answers (e.g. you can write $2^8 \cdot 4!$ instead of multiplying out to 6144.)

If you need extra space for your solutions, there is an extra page at the back of the exam. If you need extra space for any problem, write CONTINUED IN EXTRA SPACE on the page where the problem is given to you. In the extra space write the problem number that you are solving in that space.

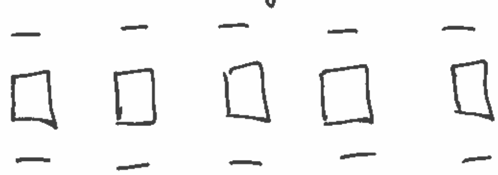
Problem 1: 100 students are training to be chess champions. The top 10 of these 100 students are chosen for the finals and then these 10 are paired up to play each other in 5 games. The students only care about whether they are chosen for the finals and who their opponent in the finals is. How many possible outcomes are there for the students who become paired up in the chess finals?

Of the 100 possible students 10 will be chosen as a subset (unordered) that make up the finalists.

The number of subsets of size 10 of a set of 100 students is $\binom{100}{10} = \frac{100!}{90!\,10!}$

Then we must add the information about which of the students will face each other.

If we imagine there are 5 chess boards set up with 1~~0~~ chair~~s~~ on each of the 2sides



there are 10 different seats for the 10 different finalists.

There are 10! ways of ordering the students into these seats.

If we switch any one board with any other moving the players with it there are no changes to the pairing.

There are 5! ways to permute the 5 boards.

If in any of the 5 boards ~~[illegible]~~ the 2 players switch sides, there is no change to the pairing. There are 2^5 ways to switch sides.

Therefore there are a total of

$$\binom{100}{10} \cdot \frac{10!}{5!\,2^5} = \frac{100!}{90!\,10!} \cdot \frac{10!}{5!\,2^5} = \boxed{\frac{100!}{90!\,5!\,2^5}}$$

possible outcomes.

Problem 2: How many subsets of $S = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are there which contain at least one of the elements 3, 6, or 9?

Let $A = \{\text{subsets containing } 3\}$
$B = \{\text{subsets containing } 6\}$
$C = \{\text{subsets containing } 9\}$

Then $|A| = |B| = |C| = 2^{11}$ because for all but 1 of the 12 elements of S, we get to choose yes or no to the question "is this element in the subset?" and for one element (3 for A, 6 for B, and 9 for C) the choice is fixed.

$A \cap B = \{\text{subsets containing 3 and containing } 6\}$
$B \cap C = \{\text{subsets containing 6 and containing } 9\}$
$A \cap C = \{\text{subsets containing 3 and containing } 9\}$

Then $|A \cap B| = |B \cap C| = |A \cap C| = 2^{10}$ for similar reasons as above but now we have two elements where the choice is fixed as yes and 10 elements with options yes or no.

$A \cap B \cap C = \{\text{subsets containing 3 and containing 6 and containing } 9\}$

$|A \cap B \cap C| = 2^9$ again for similar reasons -- now there are 3 elements (3, 6, 9) where the choice is fixed as yes and 9 elements with free choice.

By the inclusion-exclusion principle

$$
\begin{aligned}
|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= 2^{11} + 2^{11} + 2^{11} - 2^{10} - 2^{10} - 2^{10} + 2^9 \\
&= 3 \cdot 2^{11} - 3 \cdot 2^{10} + 2^9
\end{aligned}
$$

The $A \cup B \cup C$ is exactly the subsets containing at least one of the numbers 3, 6, 9 so there are $\boxed{3 \cdot 2^{11} - 3 \cdot 2^{10} + 2^9}$ such subsets

Problem 3: A children's doctor has a box of candy with 100 pieces of candy in it. Over the course of the week, the doctor sees 50 children and each of them is allowed to take candy from the box. Some children may take no candy, but each child is allowed to take as many pieces as they choose. At the end of the week if any candy is left, the receptionist takes it home to her daughter. How many different ways are there for the candy to be distributed to the 50 children in the doctor's office plus the 1 receptionist's daughter?

This is a distribution problem. We have 100 pieces of candy and 51 people to take it (including the receptionist's daughter).

If we line up the candy, we need to choose 50 split points to divide the candy of one child from the candy of the next child. Some of the split points may occur at the same position, since some children may receive no candy. The possible positions for the split points include a position before any candy and one after all the candy

Therefore we choose 50 numbers $1 \le s_1 \le s_2 \le \dots \le s_{49} \le s_{50} \le 101$ as the split points.

We define a bijection to the sets of 50 numbers satisfying $1 \le t_1 < t_2 < \dots < t_{49} < t_{50} \le 150$ defined by $t_i = s_i + i - 1$ for $i = 1, \dots, 50$

Then $(t_1, t_2, \dots, t_{50})$ are 50 distinct numbers between 1 and 150

So there are $\binom{150}{50}$ ways of choosing them.

Since the set of possible split points $1 \le s_1 \le s_2 \le \dots \le s_{49} \le s_{50} \le 101$ are in bijection with these $(t_1, t_2, \dots, t_{50})$ there are $\boxed{\binom{150}{50}}$ ways to distribute the candy.

Problem 4: There are n people working for a company. A new project will require a team of some of these people. One of the members of the team must be the team leader, and a different one ~~another~~ must be the assistant team leader. Show that the two sides of this equation count the number of possible ways of choosing the team, its leader, and assistant leader.

$$\underbrace{2^{n-2} \cdot n \cdot (n-1)}_{LHS} = \underbrace{\sum_{k=2}^{n} k(k-1)\binom{n}{k}}_{RHS}$$

For the LHS: Decision 1: Choose 1 person to be the team leader from the n people: $\binom{n}{1}$ possible choices.

Now there are $n-1$ people whose roles are undecided.

Decision 2: Choose 1 of the remaining $n-1$ people to be the assistant team leader: $\binom{n-1}{1}$ possible choices.

Now there are $n-2$ people whose roles are undecided.

For each of them there are 2 remaining options:

A: they are on the team (but not a leader or assistant leader)

B: they are not on the team

Therefore there are 2^{n-2} possible choices for what happens to the remaining $n-2$ people.

This gives $2^{n-2} \cdot n \cdot (n-1)$ (since $\binom{n}{1} = n$ and $\binom{n-1}{1} = n-1$).

For the RHS: Decision 1: Choose how many people will be on the team call this k. (this includes the leader and assistant)

There must be at least 2 people on the team to be leader & assistant

We will add up all of the options for different values of k because this is an OR decision (there are 5 people or 4 people or ...)

Decision 2: Choose the k people to be on the team from the n workers. (this will include the leader and assistant yet to be determined)

This is just choosing a subset of size k from a set of size n so there are $\binom{n}{k}$ possibilities.

Decision 3: Among the k people on the team, choose 1 team leader: $\binom{k}{1}$ ways

Decision 4: Among the remaining $k-1$ people on the team not the leader, choose one to be the assistant leader: $\binom{k-1}{1}$ ways.

In total get $\sum_{k=2}^{n} k(k-1)\binom{n}{k}$ possible ways to choose.

Problem 5: Let F_n denote the n^{th} Fibonacci number defined by the initial values $F_1 = F_2 = 1$ and the recurrence relation $F_n = F_{n-1} + F_{n-2}$.

Prove that for every $k \geq 1$, F_{4n} is divisible by 3.

Proof by induction: Base case: $n=1$

$$F_4 = F_3 + F_2 = F_2 + F_1 + F_2 = 1+1+1 = 3$$

So $F_4 = 3 = 3 \cdot 1$ is divisible by 3. ✓

Inductive hypothesis: Assume $F_{4(n-1)} = F_{4n-4}$ is divisible by 3

Then $F_{4n} = F_{4n-1} + F_{4n-2}$

$$= (F_{4n-2} + F_{4n-3}) + (F_{4n-3} + F_{4n-4})$$

$$= F_{4n-2} + 2F_{4n-3} + F_{4n-4}$$

$$= (F_{4n-3} + F_{4n-4}) + 2F_{4n-3} + F_{4n-4}$$

$$= 3 \cdot F_{4n-3} + 2 \cdot F_{4n-4}$$

Since F_{4n-4} is divisible by 3, $2 \cdot F_{4n-4}$ is divisible by 3.

$3 \cdot F_{4n-3}$ is divisible by 3 by definition of divisibility.

Therefore by factoring, $3 \cdot F_{4n-3} + 2F_{4n-4}$ is divisible by 3

Since $F_{4n} = 3 \cdot F_{4n-3} + 2 \cdot F_{4n-4}$, F_{4n} is divisible by 3.

$\boxtimes$