Homework 1

Math 145, Spring 2019

1 Counting

1. In a game, you roll one 6 sided dice, one 8 sided dice, and one 20 sided dice. How many different possible outcomes are there for the resulting roll? (We consider each dice differently so if the six sided dice rolls a 1, the eight sided dice rolls a 2, and the 20 sided dice rolls a 3, this is considered a different outcome from the situation where the six sided dice rolls a 2, the eight sided dice rolls a 1, and the 20 sided dice rolls a 3.) Explain how you got your answer in words.

   Solution: Each dice is rolled independently, so the total number of outcomes is the product of the number of outcomes for each dice. Therefore the number of possible outcomes for the rolls of all three dice is $6 \cdot 8 \cdot 20 = 960$.

2. Ten people arrive early to buy tickets for a concert. They form an ordered line. How many ways are there for the 10 different people to stand in the line? Explain how you got your answer in words.

   Solution: For the first place in line, there are 10 possible people which can stand in it. Once that person is chosen, there are 9 people left. Any of those 9 people can stand in the second place in line. Continuing, when we reach the n^{th} place in line, there are $11 - n$ people left who can stand in that spot until we reach the 10^{th} spot where there is exactly one person left to stand in the 10^{th} spot. Therefore the total number of ways the 10 people can stand in line is: $10! = 10 \cdot 9 \cdot \ldots \cdot 2 \cdot 1$.

3. Your friend has 8 shiny new quarters which are indistinguishable from each other. She flips all 8 of them and leaves them face up on the table in front of you. You cannot tell the difference between the different quarters. How many possible different resulting states can there be of the 8 quarters on the table? Explain how you got your answer in words.

   Solution: Because we cannot tell the difference between different quarters, at the end, we can only tell how many heads versus how many tails are face up on the table. If we know there are n quarters with the heads face up, there must be $8 - n$ quarters with the tails face up. Therefore the total number of possible states is the possible numbers of heads face up which can be, $0, 1, 2, \cdots, 7, 8$ so there are 9 *total possible states* for the 8 quarters on the table.

4. Alice and Bob are team captains for a soccer match. There are 12 other kids who want to play. Alice and Bob take turns choosing the remaining six kids for their team in order based on the position that will be filled. (For example, we consider the resulting states different if Alice chooses Sam as her first pick versus choosing Sam as her third pick.) How many different ways are there for all of the positions on the two teams to be filled? Explain how you got your answer in words.

   Solution: Each position is different, so this is similar to the problem of people standing in a line. For Alice's first pick there are 12 kids to choose from. For Bob's first pick there are 11 remaining kids to choose from. Alice's second pick has 10 remaining choices, and so on. Therefore the total number of different ways to fill all of the positions on both teams is $12! = 12 \cdot 11 \cdot \ldots \cdot 2 \cdot 1$.

5. Caroline and Doug are team captains for a dodgeball match. There are 10 other kids who want to play. Caroline and Doug take turns choosing kids for their team so at the end they each choose five kids for their team. The order in which kids were chosen does not affect the game. The only thing that matters is which team they are on. How many possible ways are there to split up the 10 kids into Caroline and Doug's teams? Explain how you got your answer in words.

   Solution: In this case, the order in which the kids are chosen does not affect the end state for how they play the game. Therefore, the only thing that matters is who is on Caroline's team and who is on Doug's team. We know that each team will have five kids. If we know which five kids are on Caroline's team, then we know the remaining five will be on Doug's team. Therefore we just need to count how many ways there are to choose 5 of the 10 kids to be on Caroline's team. Since the order does not matter there are

   $$\binom{10}{5} = \frac{10!}{5!5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

   possibilities. (If the order mattered there would be $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ possible ways, but then we are over counting by a factor equal to the number of reorderings of those 5 people and the number of reorderings is $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.)

6. A tennis tournament has 12 participating teams. Each match faces two teams against each other. If team A faces team B in a match, we consider this the same pairing as if team B faces team A. If team A face team B on the first court, we consider this the same pairing as if team A faces team B on the fourth court. How many ways are there to pair up the 12 teams into 6 pairings? Explain how you got your answer in words.

   Solution: First, imagine there are 6 different courts and each court has two sides. We will first count how many ways there are to assign the 12 teams to the 12 sides of the different courts: this will be $12! = 12 \cdot 11 \cdot \ldots \cdot 2 \cdot 1$–this will be an overcount of the number of pairings of the teams. One source of overcounting comes from the fact that we can take the two teams on court i and swap them both with the two teams on court j where $i, j \in \{1, 2, 3, 4, 5, 6\}$. This means we are overcounting by a factor of 6! which is the number of reorderings of the 6 courts. This gives $\frac{12!}{6!} = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7$ possible pairings, but we are still overcounting in one more way. If we look at court 1 and swap which team is on which side then we consider the resulting state to be the same. Similarly we can switch the sides of the two teams on courts $2, 3, 4, 5$, and 6. Therefore we are overcounting by a factor of 2^6. So the total number of pairings of the 12 teams is

$$\frac{12!}{6!2^6} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{2^6} = 3 \cdot 11 \cdot 5 \cdot 9 \cdot 7.$$

2 Sets

7. True or false?

   (a) x is an element of the set $\{\{x, y\}, \{z\}, 4\}$.
   False. The elements of the set are: $\{x, y\}$, $\{z\}$, and 4.

   (b) $\{\{z\}, 4\}$ is a subset of the set $\{\{x, y\}, \{z\}, 4\}$.
   True, because both elements of the subset: $\{z\}$ and 4, are elements of the set

   (c) The integers are a subset of the real numbers.
   True, because every element of the integers is an integer which is a real number.

8. Prove that if A, B, and C are sets then $(A \cup B) \cup C = A \cup (B \cup C)$.

   Solution: First, suppose $x \in (A \cup B) \cup C$. Then by definition, either $x \in (A \cup B)$ or $x \in C$. If $x \in (A \cup B)$ then by definition, $x \in A$ or $x \in B$. If $x \in A$ then $x \in A \cup (B \cup C)$. If $x \in B$ then $x \in (B \cup C)$ therefore $x \in A \cup (B \cup C)$. If $x \in C$ then $x \in (B \cup C)$ so $x \in A \cup (B \cup C)$. Therefore in all possible cases, $x \in A \cup (B \cup C)$. Therefore $(A \cup B) \cup C \subseteq A \cup (B \cup C)$.

   The other direction is similar: suppose $y \in A \cup (B \cup C)$. Then either $y \in A$ or $y \in (B \cup C)$. If $y \in A$ then $y \in (A \cup B)$ so $y \in (A \cup B) \cup C$. If $y \in (B \cup C)$ then either $y \in B$ or $y \in C$. If $y \in B$ then $y \in (A \cup B)$ so $y \in (A \cup B) \cup C$. If $y \in C$ then $y \in (A \cup B) \cup C$. Therefore $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.

   Since we have both subset inclusions, the two sets must be equal.

9. Prove that if A, B, and C are sets then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

   Solution: Suppose $x \in A \cap (B \cup C)$. Then, by definition, $x \in A$ and $x \in (B \cup C)$. Therefore $x \in A$ and either $x \in B$ or $x \in C$. If $x \in A$ and $x \in B$ then $x \in A \cap B$. If $x \in A$ and $x \in C$ then $x \in A \cap C$. Therefore if $x \in A \cap (B \cup C)$ then either $x \in (A \cap B)$ or $x \in (A \cap C)$, so $x \in (A \cap B) \cup (A \cap C)$. This shows that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

   For the other direction, suppose $x \in (A \cap B) \cup (A \cap C)$. Then either $x \in (A \cap B)$ or $x \in (A \cap C)$. If $x \in (A \cap B)$ then $x \in A$ and $x \in B$, therefore $x \in A$ and $x \in (B \cup C)$ so $x \in A \cap (B \cup C)$. If $x \in A \cap C$ then $x \in A$ and $x \in C$, therefore $x \in A$ and $x \in (B \cup C)$, so $x \in A \cap (B \cup C)$. Therefore we have shown that every element of $(A \cap B) \cup (A \cap C)$ is an element of $A \cap (B \cup C)$, i.e. $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

   Since we have both subset inclusions, the two sets must be equal.

10. There are seven gymnasts at a competition: Amanda, Dorothy, Gabby, Jacquelyn, Kelly, Madison, and Simone. One is chosen for the gold medal, one for silver, and one for bronze. What are the number of possibilities for the winning gold, silver, and bronze medalists? Explain how this problem can be understood as a question of the number of *ordered* k-element subsets of a set S with n elements. What is the set S? What are k and n?

    Solution: Any of the 7 gymnasts could receive the gold medal. Of the remaining 6, any of them could receive the silver medal. After the gold and silver medalists are chosen, any of the remaining 5 could receive the bronze medal. Therefore there are $7 \cdot 6 \cdot 5$ different possibilities for the winning gymnasts.

    This can be viewed as a question of the number of ordered $k = 3$ element subsets of a set $S = \{\text{Amanda, Dorothy, Gabby, Jacquelyn, Kelly, Madison, and Simone}\}$ where S has $n = 7$ elements.

11. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many (unordered) subsets does S have where the size of the subset (number of elements of the subset) is any *positive even number*? Explain how you got your answer in words.

    Solution: A subset of S with a positive *even* number of elements can have $2, 4, 6$, or 8 elements. The number of unordered k-element subsets of a set S with n elements is

    $$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

    Therefore the number of 2 element subsets of S is $\binom{9}{2}$, the number of 4 element subsets of S is $\binom{9}{4}$, the number of 6 element subsets is $\binom{9}{6}$, and the number of 8 element subsets is $\binom{9}{8}$. Therefore the total number of unordered subsets with a positive even number of elements is:

    $$\binom{9}{2} + \binom{9}{4} + \binom{9}{6} + \binom{9}{8} = \frac{9 \cdot 8}{2} + \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2} + \frac{9 \cdot 8 \cdot 7}{3 \cdot 2} + \frac{9}{1} = 36 + 126 + 84 + 9 = 255$$

3 Induction

12. Prove by induction that the sum of the first n squares $(1 + 4 + 9 + \cdots + n^2)$ is $n(n+1)(2n+1)/6$.

    Solution: First we check the base case when $n = 1$. In that case the sum of the first 1 square is just 1. The formula agrees with this because

    $$\frac{1(1+1)(2 \cdot 1 + 1)}{6} = 1.$$

Now, we inductively assume that the formula holds for $n-1$ so we inductively assume that

$$\begin{aligned} 1+2^2+\cdots+(n-1)^2 &= \frac{(n-1)((n-1)+1)(2(n-1)+1)}{6} \\ &= \frac{(n-1)(n)(2n-1)}{6} \end{aligned}$$

Now we want to calculate the sum of the first n squares. Using the inductive assumption we have

$$\begin{aligned} 1+2^2+\cdots+(n-1)^2+n^2 &= \frac{(n-1)(n)(2n-1)}{6}+n^2 \\ &= \frac{2n^3-3n^2+n+6n^2}{6} \\ &= \frac{2n^3+3n^2+n}{6} \\ &= \frac{n(2n^2+3n+1)}{6} \\ &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

Thus we have proven the formula by induction.

13. Prove by induction that

$$2^0+2^1+\cdots+2^{n-1}=2^n-1.$$

Solution: For the base case, $n=1$ and we have $2^0=2^1-1$ which is indeed true because $1=2-1$.

Now, we inductively assume that

$$2^0+2^1+\cdots+2^{n-2}=2^{n-1}-1.$$

Therefore

$$2^0+2^1+\cdots+2^{n-2}+2^{n-1}=2^{n-1}-1+2^{n-1}=2(2^{n-1})-1=2^n-1.$$

Thus we have proven the formula by induction.

14. Let $a_0=1$ and let $a_{m+1}=2a_m+1$ for all positive integers $m\geq 1$. Find an explicit formula for a_m (in terms of m only) and prove your formula is correct.

Solution: Writing out the first few terms:

$$a_0=1,\ a_1=2(1)+1=3,\ a_2=2(2(1)+1)+1=2(3)+1=7,$$

$$a_3=2(2(2(1)+1)+1)+1=2(2(3)+1)+1=2(7)+1=15$$

Using this pattern, we conjecture that

$$a_m = 2^m + 2^{m-1} + \cdots 2^1 + 2^0$$

and using the previous problem, we guess that $a_m = 2^{m+1} - 1$.

Now to prove this guess, by induction, we first check the base case when $m = 0$. In this case $a_0 = 1 = 2 - 1 = 2^{0+1} - 1$.

Now inductively assume that $a_{m-1} = 2^m - 1$. Then

$$a_m = 2a_{m-1} + 1 = 2(2^m - 1) + 1 = 2^{m+1} - 2 + 1 = 2^{m+1} - 1.$$

So we have proven our predicted formula by induction.

15. Suppose you have a square piece of paper, and after 1 minute you cut it into four squares. After the second minute you cut one of those squares into four squares. Every minute you cut one square of paper into four squares. Prove that after n minutes, the number of squares you have is $3n + 1$.

Solution: First we check the base case when $n = 0$, meaning 0 minutes have passed. This means we only have 1 square of paper which we have not cut at all. Since $1 = 3(0) + 1$ this verifies the base case.

Next, we inductively assume that after $n - 1$ minutes, the number of squares of paper you have is $3(n - 1) + 1$. Now after the n^{th} minute, we cut one of these squares into four squares. Then we have $3(n - 1) + 1 - 1$ squares from one minute ago that are unchanged, plus four more squares that have been produced from the last square from one minute ago. Therefore there are a total of

$$3(n - 1) + 1 - 1 + 4 = 3n - 3 + 4 = 3n + 1$$

squares.