Homework 4

Math 145, Spring 2019

Every solution must contain an explanation written in words supporting your numerical solution to receive credit.

1 Pascal's Triangle

1. On a grid of points placed on the integer points of the plane, we call a path a *staircase path* if as you move from the start point to the end point, every step moves either *up* one unit or *right* one unit.

   (a) How many staircase paths are there from $(0,0)$ to $(13,9)$? Justify your answer.

   (b) How many staircase paths are there from $(0,0)$ to $(9,13)$? Justify your answer.

   (c) How many staircase paths are there from $(2,3)$ to $(6,9)$? Justify your answer.

   (d) How many staircase paths are there from $(0,0)$ to $(8,12)$ which *do not* pass through $(4,2)$? Justify your answer.

2. Prove **by induction** that for any $k, n \geq 0$,

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}.$$

3. Write out the first 15 rows of Pascal's triangle (rows 0 through 14) over the integers mod 2. (All of your entries should be either 0 or 1 with the rules $0+0=0$, $1+0=1$, $0+1=1$ and $1+1=0$.) You can color the regions with 0's if you'd like to see a pretty pattern.

   Use this to list out which of the numbers $\binom{n}{k}$ are *even* numbers when $0 \leq k \leq n \leq 14$. (Don't forget that the first entry in each row corresponds to $k=0$ not $k=1$, so count carefully.)

2 More Counting Problems

4. At a celebration, they plan to drop colored balloons from the ceiling at midnight. The net containing the balloons holds exactly 50 balloons. The organizers have decided they would like the balloons to be an assortment of the colors blue, gold, silver, and purple. They would like at least 5 balloons of each color. How many different ways are there for the assortment of balloons to be chosen to satisfy the organizers' requirements?

5. Blaise, Carl, Henri, Gottfried, Isaac, and Leonhard are working together on a project which has turned up 100 different problems. None of them has enough time to finish the entire project, but they agree to work on some of them based on how much time they have as follows:

   - Blaise will solve 8 problems
   - Carl will solve 9 problems
   - Henri will solve 11 problems
   - Gottfried will solve 5 problems
   - Isaac will solve 5 problems
   - Leonhard will solve 10 problems

   The problems are all different from each other. How many different ways are there to assign problems to these six people in the quantity that they agreed to solve?

6. We distribute n (identical) pennies to k students.

   (a) First, we require each of the k students should receive at least 1 penny each. How many ways are there to distribute the n pennies?

   (b) Next, we do not require that each of the k students must receive at least 1 penny, they just will each be given some number ≥ 0 of pennies. How many ways are there to distribute the n pennies?

   (c) Finally, we insist that students who got an A grade must get at least 1 penny, but any of the other students may get no pennies. If ℓ of the students received an A grade, and the other $k - \ell$ did not receive an A grade, how many different ways are there to distribute the n pennies?

7. Consider the equation
$$a + b + c + d + e = 31.$$

   (a) If $a, b, c, d, e \geq 1$ are each *positive integers* (≥ 1), how many possible solutions (a, b, c, d, e) are there to the equation?

   (b) If $a, b, c, d, e \geq 0$ are each *non-negative integers*, how many possible solutions (a, b, c, d, e) are there to the equation?

3 Fibonacci Numbers

In this section F_i denotes the i^{th} Fibonacci number defined by the initial terms $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

8. How many subsets does the set $\{1, 2, \ldots, n\}$ have that contain no two consecutive integers? Justify your answer.

9. Prove F_{5n} is divisible by 5.

10. Prove for all $n \geq 1$
$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}.$$
11. Prove for all $n \geq 1$
$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n \cdot F_{n+1}.$$
12. Prove for all $n \geq 2$
$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$