Homework 7

Math 145, Spring 2019

Every solution must contain an explanation written in words supporting your numerical solution to receive credit.

1. Consider the Prüfer code 22641. Complete this to a 2 row code giving the edges and then draw the corresponding tree. You can check your work by starting with your answer tree and writing down the Prüfer code to make sure it is the one you started with.

2. What tree gives the Prüfer code $000\cdots 0$ with $n-2$ consecutive 0's? What tree gives the Prüfer code $111\cdots 1$ with $n-2$ consecutive 1's (for $n \geq 2$)?

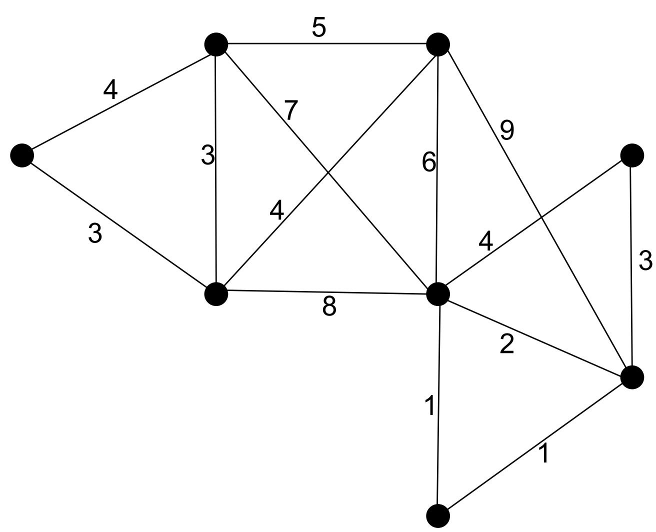
3. Show that the following two rows could not be the output of the ordered edges from the Prüfer algorithm (before deleting the top row and last 0 in the bottom row).

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 3 & 5 & 5 & 2 & 0 \end{matrix}$$

   What graph has these edges?

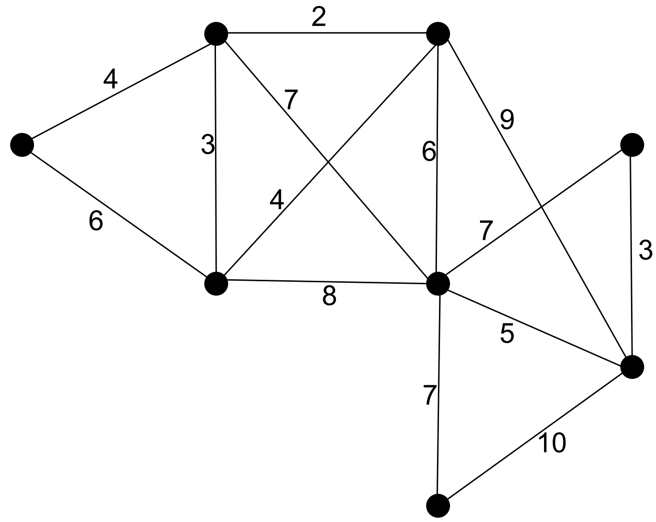
4. Follow Kruskal's greedy algorithm to find the spanning trees of minimal cost and the total cost for those spanning trees in the following weighted graphs (the graphs are the same but the weights are different):

   (a) G_1



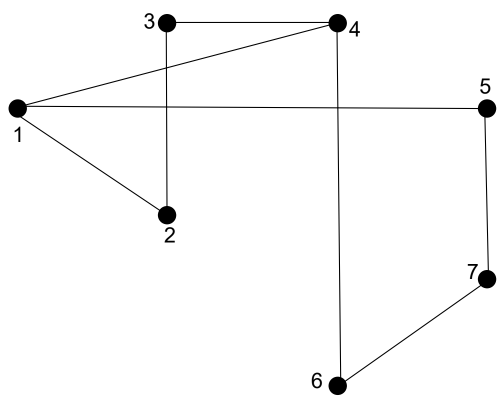
   Continue for part (b) on the next page.

(b) G_2



5. In the following problems, recall that the *adjacency matrix* (or incidence matrix) for a simple graph with n vertices is an $n \times n$ matrix with entries that are all 0 or 1. The entries on the diagonal are all 0, and the entry in the i^{th} row and j^{th} column is 1 if there is an edge between vertex i and vertex j and is 0 if there is not an edge between vertex i and vertex j.

   (a) Write down the adjacency matrix for the following simple graph with the vertices ordered according to the labels in the pictures.



   (b) Let A be the adjacency matrix from the previous example. Calculate the matrix A^2 using matrix multiplication. Check that in this example the (i, j) entry of the matrix A^2 is exactly the number of length 2 walks from vertex i to vertex j.

   (c) Prove that for any simple graph with adjacency matrix A, the (i, j) entry of the matrix A^2 gives the number of walks of length 2 from i to j.

   (d) **Challenge:** Prove by induction that the (i, j) entry of A^k gives the number of length k walks from vertex i to vertex j.

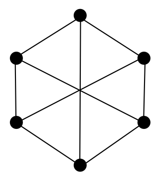
   Corollary: A simple graph with n vertices is connected if and only if every entry except the diagonal is positive in the matrix

$$A + A^2 + A^3 + \cdots + A^{n-1}.$$

6. Let G be a forest, i.e. a graph with no cycles. Suppose G has n vertices. Show that if G has k connected components, then G has $n-k$ edges.

7. What is the maximal number of edges a simple graph can have if it has n vertices and it is not connected?

8. Determine whether the following graphs have planar embeddings or not and prove it.

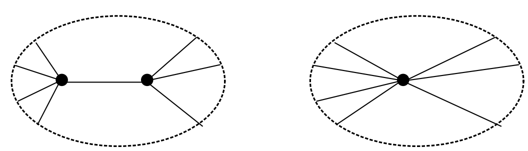
   (a) This graph: 

   (b) K_4

   (c) K_6

   (d) All trees

9. Recall that a *contraction* of a graph G along an edge e squeezes that edge to a point, identifying the vertices at its two endpoints as in this picture:



   (a) Show that the contraction of any tree along any edge is still a tree.

   (b) Use this to give a new proof by induction that a tree with n vertices has $n-1$ edges.