Quiz

Math 145, Spring 2019

Name: Solutions

Problem 1: Let S be the set $\{1, 2, 3, 4, 5, 6\}$.

(a) How many subsets of S are there which contain 5? Prove and explain your answer in words.

Every subset of S containing 5 is a subset of

$\{1,2,3,4,6\}$ union $\{5\}$.

The number of subsets of $\{1,2,3,4,6\}$ is 2^n

where $n = |\{1,2,3,4,6\}| = 5$

Therefore there are 2^5 subsets of S which contain 5.

(b) How many subsets of S are there of size 3 (i.e. subsets with exactly 3 elements)? Prove and explain your answer in words.

A subset is unordered and has nonrepeating elements

so it is a dependent unordered problem.

There are 6 elements in S to choose from and

we want to choose exactly 3 of these 6 elements.

Therefore the number of subsets of size 3 is

$\binom{6}{3} = \frac{6!}{3!\,3!}$ ($\frac{6!}{3!}$ is the number of ordered subsets of size 3 and $3!$ is the number of reordering

Problem 2: There are 15 cities you would like to visit, but over summer break you only have time to visit 5. How many different ways are there to visit 5 of these cities and then return to Davis (different orders count as different ways because your routes will be different)? Prove and explain your answer in words.

This is an ordered problem and since you do not want to visit the same city again after visiting it once, this is a dependent problem.

There are 15 · choices for the 1^{st} city you visit
14 · choices for the 2^{nd} city
13 · " 3^{rd}
12 · " 4^{th}
11 " 5^{th}

So in total there are $15 \cdot 14 \cdot 13 \cdot 12 \cdot 11$ possible trips you could take.

Problem 3: Prove that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Let $x \in A \setminus (B \cup C)$

Then $x \in A$ and $x \notin B \cup C$

Therefore $x \in A$ and $x \notin B$ and $x \notin C$

$\Rightarrow$ $x \in A$ and $x \notin B$ and $x \in A$ and $x \notin C$

$\Rightarrow$ $x \in A \setminus B$ and $x \in A \setminus C$

$\Rightarrow$ $x \in (A \setminus B) \cap (A \setminus C)$

So $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$.

Conversely, let $y \in (A \setminus B) \cap (A \setminus C)$.

Then $y \in A \setminus B$ and $y \in A \setminus C$

$\Rightarrow$ $y \in A$ and $y \notin B$ and $y \in A$ and $y \notin C$.

$\Rightarrow$ $y \in A$ and $y \notin B$ and $y \notin C$.

$\Rightarrow$ $y \in A$ and $y \notin B \cup C$

$\Rightarrow$ $y \in A \setminus (B \cup C)$

So $(A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C)$.

$\square$

Problem 4: Prove by induction that the sum of the odd numbers

$$1 + 3 + 5 + \cdots + (2n-3) + (2n-1) = n^2.$$

Let's prove this by induction.

Base case: $n=1$ so $2n-1=1$ so the left hand side is just 1

the right hand side is $1^2 = 1$

so $LHS = 1 = 1 = RHS$ ✓.

Inductive hypothesis: Inductively assume

$$1+3+5+\ldots+(2(n-1)-3)+(2(n-1)-1) = (n-1)^2$$

multiplying things out $\Rightarrow 1+3+5+\ldots+(2n-5)+(2n-3) = n^2-2n+1$

adding $2n-1$ to both sides $\Rightarrow 1+3+5+\ldots+(2n-5)+(2n-3)+(2n-1) = n^2 \underbrace{-2n+1+(2n-1)}_{\text{cancel out} \to 0}$

$\Rightarrow 1+3+5+\ldots+(2n-5)+(2n-3)+(2n-1) = n^2$

So the formula holds for all $n \geq 1$.

Problem 5: For $0 \leq k \leq n$, we defined n choose k by the formula

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

It is not obvious from the definition that this is an integer.

Prove that $\binom{n}{k}$ is an integer by induction on n.

To do this, it will be very helpful if you first prove (without induction just arithmetic) the formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

(You can use the formula without proving it for partial credit.)

First we prove this formula:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(n-k)!(k-1)!} + \frac{(n-1)!}{(n-k-1)!\,k!}$$

Finding a common denominator $= \dfrac{k\cdot(n-1)!}{(n-k)!\,k!} + \dfrac{(n-k)(n-1)!}{(n-k)!\,k!}$

$$= \frac{(k+n-k)(n-1)!}{(n-k)!\,k!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}.$$

Now our inductive statement we want to prove is:

$\binom{n}{k}$ is an integer for all $1 \leq k \leq n$.

Base case: $n=1$. Then the options for k are only $k=0$ or 1.

$\binom{1}{0} = \binom{1}{1} = \frac{1!}{0!\,1!} = \frac{1}{1} = 1$ is an integer. ✓

Inductive hypothesis: Now suppose $\binom{n-1}{k}$ is an integer for all $1 \leq k \leq n-1$.

Then $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ is the sum of 2 integers whenever $k \leq n-1$, so in these cases $\binom{n}{k}$ is an integer.

For the last case when $k=n$ we check directly

$\binom{n}{n} = \frac{n!}{0!\,n!} = \frac{n!}{n!} = 1$ is an integer.

Extra Space: