Boundary of a subset

Theorem 1. *Let (X, τ) be a topological space and $K \subseteq X$ a subset. Let $Bdry(K) = \overline{K} \cap \overline{(X \setminus K)}$. Then $Bdry(K) = \overline{K} \setminus Int(K)$.*

Proof. We want to prove that $\overline{K} \cap \overline{(X \setminus K)} = \overline{K} \setminus Int(K)$.

Step 1: We will show $\overline{K} \cap \overline{(X \setminus K)} \subseteq \overline{K} \setminus Int(K)$.

By properties of intersection $\overline{K} \cap \overline{(X \setminus K)} \subset \overline{K}$. Thus, we just need to show that no point in $\overline{K} \cap \overline{(X \setminus K)}$ is in $Int(K)$.

Suppose $x \in Int(K)$. Then by definition of the interior there exists an open set U such that $x \in U \subset K$. Therefore $x \in U$, U is open, and $U \cap (X \setminus K) = \emptyset$. Therefore x is not a limit point of $X \setminus K$ and $x \notin X \setminus K$. Therefore $x \notin \overline{(X \setminus K)}$.

Step 2: We will show $\overline{K} \setminus Int(K) \subseteq \overline{K} \cap \overline{(X \setminus K)}$.

Suppose $x \in \overline{K} \setminus Int(K)$. Then $x \in \overline{K}$ and we want to show that $x \in \overline{(X \setminus K)}$. Since $x \notin Int(K)$, there is no open set U such that $x \in U$ and $U \subset K$. This means that for every open set U such that $x \in U$, U is not a subset of K, i.e. $U \cap (X \setminus K) \neq \emptyset$. This precisely means that x is a limit point of $X \setminus K$. Therefore $x \in \overline{(X \setminus K)}$.

$\square$