Homework 7

Math 147, Fall 2018

1. Suppose (X, τ_X) and (Y, τ_Y) are topological spaces. Let $(X \times Y, \tau_{X\times Y})$ be the product of X with Y with the product topology. Define an equivalence relation as follows: if $(x_1, y_1), (x_2, y_2) \in X \times Y$ then $(x_1, y_1) \sim (x_2, y_2)$ if and only if $y_1 = y_2$. Let $(X \times Y)/_\sim$ be the quotient of $X \times Y$ by this equivalence relation, and put the quotient topology on this space. Show that with the quotient topology, $(X \times Y)/_\sim$ is homeomorphic to (X, τ_X) by defining a map $f : (X \times Y)/_\sim \to Y$, which is continuous and has continuous inverse.

2. Suppose (X, τ_X) and (Y, τ_Y) are topological spaces and suppose $f : (X, \tau_X) \to (Y, \tau_Y)$ is a *continuous surjective* map. Consider the equivalence relation on X defined by $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$. Let $(X/_\sim, \tau_q)$ be the quotient space of X by this equivalence relation with the quotient topology.

   Define a map $\tilde{f} : (X/_\sim, \tau_q) \to (Y, \tau_Y)$ by setting $\tilde{f}([x]) = f(x)$ where $[x]$ denotes the equivalence class containing the point x.

   (a) Show that $\tilde{f}$ is well defined (show that if $x_1 \sim x_2$ then $\tilde{f}([x_1]) = \tilde{f}([x_2])$).

   (b) Show that $\tilde{f}$ is continuous.

   (c) Show that $\tilde{f}$ has an inverse $\tilde{f}^{-1}$ such that $\tilde{f} \circ \tilde{f}^{-1}(y) = y$ and $\tilde{f}^{-1}(\tilde{f}([x])) = [x]$.

   (d) Show that there is an example where $\tilde{f}^{-1}$ is not continuous. (Hint: show that if $X = \mathbb{R}^2$ and τ_X is the *discrete topology* and $Y = \mathbb{R}$ with τ_Y being the Euclidean topology and we define f by $f((x_1, x_2)) = x_1$ then $\tilde{f}^{-1} : (\mathbb{R}, \tau_{Euc}) \to (X/_\sim, \tau_q)$ is not continuous)

3. Prove that if $x \in \mathbb{R}$ then $\mathbb{R} \setminus \{x\}$ is not connected.

4. Let A and B be subsets of a topological space (X, τ) such that (A, τ_A) and (B, τ_B) are connected with the subspace topology. If $A \cap B \neq \emptyset$ show that $A \cup B$ with the subspace topology is connected.

5. Prove that if (X, τ) has n connected components $C_1, \cdots, C_n$ and $X = C_1 \cup \cdots \cup C_n$ then there is a *surjective* continuous map $f : (X, \tau) \to \{1, 2, \cdots, n\}$ where $\{1, 2, \cdots, n\}$ has the discrete topology. Conversely if there is a *surjective* continuous map $f : (X, \tau) \to \{1, 2, \cdots, n\}$ (again using the discrete topology on $\{1, 2, \cdots, n\}$) prove that (X, τ) has *at least* n disjoint open and closed subsets.