Practice Midterm 2

Math 147, Fall 2018

Name:

Problem 1: Consider the subset $Y = [-1, 1]$ of the real line $\mathbb{R}$. Let $\mathbb{R}$ have the Euclidean topology τ_{Euc} and let τ_Y denote the subspace topology on Y.

(a) Is $A = \{x \mid \frac{1}{2} < |x| \leq 1\}$ open in τ_Y? Prove it is or is not.

A is open in (Y, τ_Y) because $A = ((-\frac{3}{2}, -\frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2})) \cap Y$ and $(\frac{1}{2}, \frac{3}{2})$ is an open ball of radius $\frac{1}{2}$ around 1 so it is open in $(\mathbb{R}, \tau_{Euc})$ and similarly $(-\frac{3}{2}, -\frac{1}{2})$ is the open ball of radius $\frac{1}{2}$ around -1 in $(\mathbb{R}, \tau_{Euc})$ so since the union of open sets is open, we have written A as the intersection of an open subset of X with Y.

(b) Is $B = \{x \mid \frac{1}{2} \leq |x| < 1\}$ open in τ_Y? Prove it is or is not.

B is not open in τ_Y. If it were then $B = U \cap Y$ for $U \subset \mathbb{R}$ where $U \in \tau_{Euc}$. Therefore $\frac{1}{2} \in U$. Since U is open in the Euclidean topology on $\mathbb{R}$, there exists $\varepsilon > 0$ such that $B_\varepsilon(\frac{1}{2}) \subseteq U$. Therefore $(\frac{1}{2} - \varepsilon, \frac{1}{2} + \frac{\varepsilon}{)} \subseteq U$. Therefore $\max\{0, \frac{1}{2} - \frac{\varepsilon}{2}\} \in U$, but $\max\{0, \frac{1}{2} - \frac{\varepsilon}{2}\} \in Y$ so $\max\{0, \frac{1}{2} - \frac{\varepsilon}{2}\} \in U \cap Y = B$, but B does not contain any elements less than $\frac{1}{2}$, so we have a contradiction.

Problem 2: Let $X = \mathbb{R}^2$ with the Euclidean topology. Define an equivalence relation $\sim$ on X by $(x_1, x_2) \sim (z_1, z_2)$ if and only if $x_1^2 + x_2^2 = z_1^2 + z_2^2$. Let $Y = [0, \infty)$ with the Euclidean topology. Construct a map $f : X/_\sim \to Y$ and show that f is *well-defined, continuous and has an inverse*. You do NOT need to prove that f^{-1} is continuous (f^{-1} probably will be continuous, you just do not need to prove it).

Let $f : X/_\sim \to Y$ be defined by

$$f([(x_1, x_2)]) = \sqrt{x_1^2 + x_2^2}$$

f **is well defined:** Suppose $[(x_1, x_2)] = [(z_1, z_2)]$, then $x_1^2 + x_2^2 = z_1^2 + z_2^2$ so

$$f([(x_1, x_2)]) = \sqrt{x_1^2 + x_2^2} = \sqrt{z_1^2 + z_2^2} = f([(z_1, z_2)])$$

f **is continuous:** Let $p : X \to X/_\sim$ be the quotient map $p((x_1, x_2)) = [(x_1, x_2)]$. Then the composition $f \circ p : \mathbb{R}^2 \to [0, \infty)$ is the map $f((x_1, x_2)) = \sqrt{x_1^2 + x_2^2}$ which is a continuous map from $\mathbb{R}^2$ to $\mathbb{R}$ between Euclidean spaces by calculus arguments. Therefore for any open subset $U \subset \mathbb{R}$, $(f \circ p)^{-1}(U)$ is open in $\mathbb{R}^2$.

In the quotient topology, $f : X/_\sim \to Y$ is continuous if and only if $f \circ p : X \to Y$ is continuous, so f is continuous.

f **has an inverse:** Define $f^{-1} : Y \to X/_\sim$ by

$$f^{-1}(r) = [(r, 0)]$$

Then we will show that $f(f^{-1}(r)) = r$ and $f^{-1}(f([(x_1, x_2)])) = [(x_1, x_2)]$.

For the first statement: if $r \in [0, \infty)$,

$$f(f^{-1}(r)) = f([(r, 0)]) = \sqrt{r^2 + 0^2} = r$$

For the second statement: if $[(x_1, x_2)] \in X/_\sim$,

$$f^{-1}(f([(x_1, x_2)])) = f^{-1}(\sqrt{x_1^2 + x_2^2}) = [(\sqrt{x_1^2 + x_2^2}, 0)]$$

$$[(x_1, x_2)] = [(\sqrt{x_1^2 + x_2^2}, 0)]$$

because $(x_1, x_2) \sim (\sqrt{x_1^2 + x_2^2}, 0)$ because $\sqrt{x_1^2 + x_2^2} = \sqrt{(\sqrt{x_1^2 + x_2^2})^2 + 0^2}$.

Problem 3: Let τ_{Zar} be the Zariski topology on $\mathbb{R}$. Remember that a set $U \subset \mathbb{R}$ is open in the Zariski topology if and only if $\mathbb{R} \setminus U$ is finite or $U = \emptyset$. Prove that $(\mathbb{R}, \tau_{Zar})$ is not Hausdorff.

Consider $1, 2 \in (\mathbb{R}, \tau_{Zar})$. These are distinct points in $\mathbb{R}$. Suppose for contradiction that $(\mathbb{R}, \tau_{Zar})$ were Hausdorff. Then there would be open sets $U_1, U_2 \in \tau_{Zar}$ such that $1 \in U_1$ and $2 \in U_2$ and $U_1 \cap U_2 = \emptyset$.

Since $U_1, U_2 \in \tau_{Zar}$ and they are both non-empty, $X \setminus U_1$ and $X \setminus U_2$ are both finite sets.

Since $U_1 \cap U_2 = \emptyset$, we get that $X \setminus (U_1 \cap U_2) = X \setminus \emptyset = X$.

By DeMorgan's laws, $X = X \setminus (U_1 \cap U_2) = (X \setminus U_1) \cup (X \setminus U_2)$. But since $X \setminus U_1$ and $X \setminus U_2$ are finite sets, their union is finite which would imply X is finite, but $X = \mathbb{R}$ which has infinitely many points so we get a contradiction.

Problem 4: Suppose $f : (X, \tau) \to (\{1, 2, 3\}, \tau_{dis})$ is a continuous *surjective* function (τ_{dis} is the discrete topology). Prove that there are non-empty closed subsets, A, B, and C of X such that $X = A \cup B \cup C$ and $A \cap B = A \cap C = B \cap C = \emptyset$.

Let $A = f^{-1}(\{1\})$, $B = f^{-1}(\{2\})$ and $C = f^{-1}(\{3\})$.

In the discrete topology, $\{1\}$, $\{2\}$, and $\{3\}$ are open sets because every subset is open. They are also closed sets because their complements are open since every subset is open. Therefore A, B and C are each both open and closed because f is continuous. A, B, and C are each non-empty because f is surjective. A, B, and C are disjoint because f is a function so we cannot have a point $x \in A \cap B$ because then $f(x) = 1$ and $f(x) = 2$ which cannot happen if f is a function. Similarly $B \cap C = \emptyset$ and $A \cap C = \emptyset$.