EMBEDDING ABSTRACT MANIFOLDS IN $\mathbb{R}^M$

MATH 180, WINTER 2023

As you research, you may find more examples, definitions, and questions, which you definitely should feel free to include, but make sure you at least go through the following discussion and questions.

(1) Find the topological definition of *compact*, which does not depend on an embedding into $\mathbb{R}^M$ or a metric, but only involves open sets. Show that any compact manifold has an atlas with finitely many coordinate charts.

(2) If X is a compact manifold which is k-dimensional, with an atlas that contains N coordinate charts $\{\phi_i : U_i \to V_i\}_{i\in\{1,\dots,N\}}$, construct an embedding of X into $\mathbb{R}^{kN}$ as follows:

Look at the project on building manifolds abstractly and transition functions. Understand the definition of transition functions $\phi_{i,j} : V_{i,j} \to V_{j,i}$ that is written there. In that project, it is proven that there is a homeomorphism

$$F : X \to \left(\sqcup_{i=1}^{N} V_i\right) / \sim \qquad \text{where } (x_1, \cdots, x_k) \sim \phi_{i,j}(x_1, \cdots, x_k) \text{ for all } (x_1, \cdots, x_k) \in V_{i,j}$$

using the transition functions $\phi_{i,j}$. You may assume this homeomorphism exists.

Next, define an embedding $A_i : V_i \to \mathbb{R}^{kN}$ by

$$A_i(x_1, \cdots, x_k) = (\phi_{i,1}(x_1, \cdots, x_k), \cdots, \phi_{i,N}(x_1, \cdots, x_k)) .$$

Note that $\phi_{i,j}(x_1, \cdots, x_k)$ is a point in $\mathbb{R}^k$ so it has k coordinates, therefore there are a total of kN coordinates in the output of the function A_i. Also note that $\phi_{i,i}$ is the identity map (check why this is true using the definition of the transition functions).

Now prove that $A_i(x_1, \cdots, x_k) = A_j(\phi_{i,j}(x_1, \cdots, x_k))$.

Use this to show that you have a well defined map

$$B : \left(\sqcup_{i=1}^{N} V_i\right) / \sim \to \mathbb{R}^{kN}$$

given by $B([x]) = A_i(x)$ for $x \in V_i$.

Verify that B is continuous.

Show that B is injective.

Now put these together to get an embedding: $B \circ F : X \to \mathbb{R}^{kN}$.

(3) Do some research about the *Whitney embedding theorem*. What is the smallest number M such that every k-dimensional manifold can embed into $\mathbb{R}^M$? What is the "weak Whitney embedding theorem"?

(4) For a 1-dimensional manifold, if you have N charts, the above construction will give you an embedding $B \circ F : X \to \mathbb{R}^N$. There is a projection $\pi : \mathbb{R}^N \to \mathbb{R}^{N-1}$ which projects to the first $N-1$ coordinates.

Draw a picture of an embedding of a compact 1-dimensional manifold embedded in $\mathbb{R}^3$ (let $K \subset \mathbb{R}^3$ be the image of the embedding) such that applying the projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ $\pi(x, y, z) = (x, y)$ to K is NOT injective–draw $\pi(K) \subset \mathbb{R}^2$. Look up the "Whitney trick." Apply the Whitney trick to your picture of $\pi(K) \subset \mathbb{R}^2$ until $\pi(K)$ becomes an embedded manifold.

Now consider a situation where your embedding of the 1-manifold is into $\mathbb{R}^4$, so we have $K \subset \mathbb{R}^4$. If you did a projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ and apply this to K, what could $\pi(K)$ look like if π is not injective on K? Can you find a way to push/adjust/stretch K a little bit to make π injective on K? Can you find an explanation for why, if $K \subset \mathbb{R}^M$ and $M > 3$, that you can always push K a little bit to ensure that the projection $\pi : \mathbb{R}^M \to \mathbb{R}^{M-1}$ is injective on K? Applying this iteratively, eventually you can get to an embedding of your 1-manifold into $\mathbb{R}^3$ and then use the Whitney trick to embed it into $\mathbb{R}^2$.

(5) What are examples of 2-dimensional manifolds which can be embedded in $\mathbb{R}^2$? What are examples of 2-dimensional manifolds which cannot be embedded in $\mathbb{R}^2$? Can you describe/draw an embedding of these in $\mathbb{R}^3$? What are examples of 2-dimensional manifolds which cannot be embedded in $\mathbb{R}^3$? Can you describe an embedding of these in $\mathbb{R}^4$?