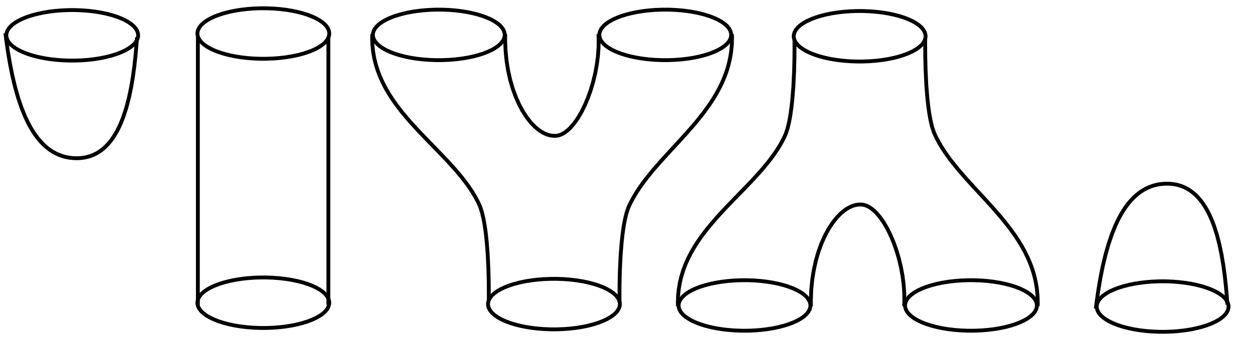
BUILDING SURFACES

MATH 180, WINTER 2023

As you research, you may find more examples, definitions, and questions, which you definitely should feel free to include, but make sure you at least go through the following discussion and questions.



These are each compact 2-manifolds with boundary. From left to right they are called the cup, cylinder, upside-down pair of pants, pair of pants, and cap.

(1) Using pictures, explain how each of the pictured 2-manifolds with boundary is homeomorphic to a subset of $\mathbb{R}^2$. Describe the subset of $\mathbb{R}^2$ explicitly for each shape, ideally with equations. Draw pictures showing how to deform the shapes that are drawn into your chosen subset of $\mathbb{R}^2$.

(2) Next, let's show that some ways of fitting together these pieces does changes the union of two pieces into something homeomorphic to just one piece:

   (a) For circle component on the boundary of any of your manifolds with boundary (you can use your homeomorphic model in $\mathbb{R}^2$), show that there is an open subset of the circle in the manifold with boundary which is homeomorphic to an annulus

$$A = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 < 4\}$$

   (b) Show that two annuli with different radii are homeomorphic. More precisely, prove that if $0 < c_1 < c_2$ and

$$A' = \{(x, y) \in \mathbb{R}^2 \mid c_1^2 \leq x^2 + y^2 < c_2^2\}$$

   then A' is homeomorphic to A.

   (c) Prove that if you pick any of the above 2-manifolds with boundary, X and glue a cylinder C to it along a single circle S^1 using a homeomorphism f, then $X \cup_f C$ is homeomorphic to X.

(d) Prove that if you glue a pair of pants (right side up or upside down) to a cup or cap, the resulting manifold with boundary is homeomorphic to a cylinder.

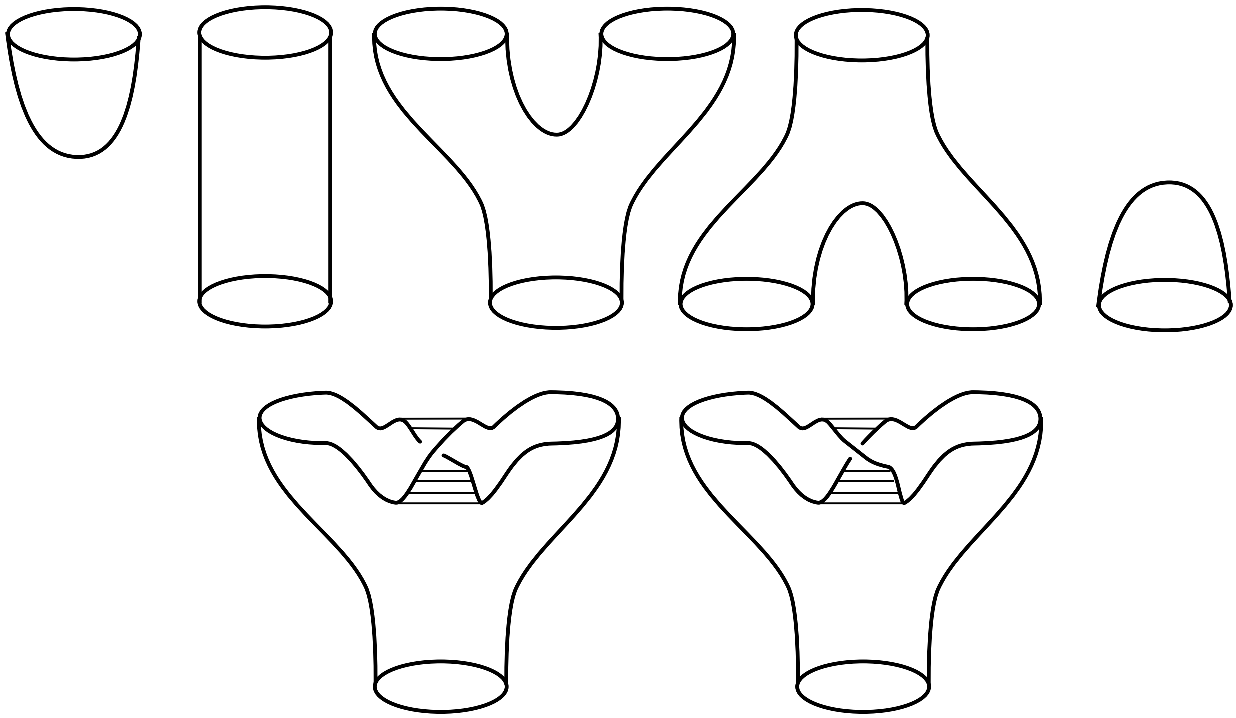
(3) Give a complete description of all possible ways to build a surface homeomorphic to a sphere using these pieces.
   (a) Find a way to label the different ways, and determine the total number of cups, caps, cylinders, pairs of pants, and upside-down pairs of pants that you use.
   (b) Describe some explicit examples that represent some of the (infintely many) possibilities.
   (c) For each case, calculate

$$\#\text{cups} + \#\text{caps} - \#\text{pair of pants} - \#\text{upsidedown pair of pants}$$

   (d) Use your work in the previous question to prove that the different ways you fit together these pieces, do all form the same homeomorphic shape.

(4) A genus g surface is a connected sum of g tori. The torus is a genus 1 surface.
   (a) For each genus g surface find one way to build it by gluing together the basic pieces we are working with.
   (b) Now find at least 4 more ways to build each genus g surface. Find at least one way which has a different number of pairs of pants than your first example, one way which has a different number of upside-down pairs of pants, one way which has a different number of cups, and one way which has a different number of cylinders.
   (c) For each case, calculate

$$\#\text{cups} + \#\text{caps} - \#\text{pair of pants} - \#\text{upsidedown pair of pants}$$



(5) Here is an extended list of surfaces with boundary.

  (a) What kind of surfaces are the new pieces? (Can you describe them in terms of some of the surfaces or surfaces with boundary that we have looked at in class?)

  (b) What new surfaces can you build with these new pieces by gluing? Give some examples. Are you always able to fit these pieces together as embedded in $\mathbb{R}^3$? Why or why not?