COVERING MAPS

MATH 180, WINTER 2023

As you research, you may find more examples, definitions, and questions, which you definitely should feel free to include in your notes and presentation, but make sure you at least answer the following questions.

A covering map $f : X \to Y$ is a continuous mapping such that for every point $y \in Y$, there is an open set $V \subseteq Y$ with $y \in V$ such that $\pi^{-1}(V)$ is a disjoint union $U_1 \sqcup \cdots \sqcup U_d$ (d may be infinite if X is non-compact) such that restricting f to each U_i gives a homeomorphism $f|_{U_i} : U_i \to V$. We say f is a d-fold covering map.

(1) Fix an integer $d \neq 0$. Consider the map $f_d : S^1 \to S^1$ defined by

$$f_d(\cos\theta, \sin\theta) = (\cos(d\theta), \sin(d\theta)).$$

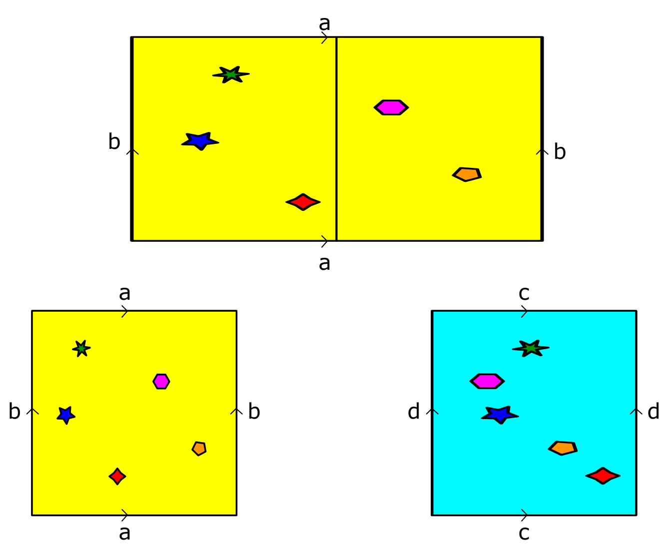
Prove that f_d is a $|d|$-fold covering map.

(2) Let $f : \mathbb{R} \to S^1$ be the map

$$f(t) = (\sin(2\pi t), \cos(2\pi t)).$$

Prove that f is an infinite order covering map.

(3) Now consider the map $f : T^2 \to T^2$ indicated in the following picture. (Remember opposite sides are glued together as indicated by the labels.)



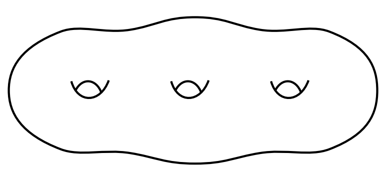
The map stretches the square by a factor of 2 in the horizontal direction and then maps the left half to the square by translation and the right half to the square by translation.

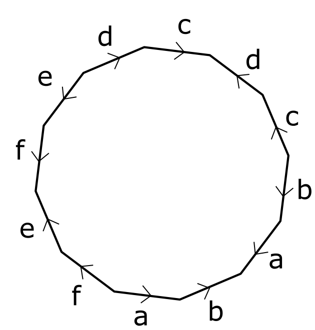
(a) According to this definition of f, a point on the center vertical line where the stretched square is cut are sent somewhere according to the left half and potentially somewhere else according to the right half. In order for f to be well-defined, these two places where f sends such a point should be the same (equivalent) in the target torus. Additionally, points which start on the boundary of the square are equivalent to other points on the boundary of the square. To say f is well-defined, we also need to check that all representatives of the same point get sent by f to points which are equivalent in the target torus. Perform these checks to verify that f is well-defined.

(b) Now show that f is a 2-fold covering map. Note that your argument to find the V, U_1, U_2 will be a bit different when you require V to contain a point y where (1) y is in the interior of the square representing T^2, (2) y is on one of the edges of the square in the target T^2, or (3) y is a vertex of the square in the target T^2. Cases (2) and (3) may be slightly more complicated arguments than case (1).

(c) For $d > 2$, find a map $T^2 \to T^2$ which is a d-fold covering map.

(4) The genus 3 surface can be represented as a subspace of $\mathbb{R}^3$ or as a polygonal representation as in the below picture.





The genus 2 surface can be represented similarly as a subspace of $\mathbb{R}^3$ or with a polygonal representation with 8 sides. Let X denote the genus 3 surface and Y denote the genus 2 surface.

(a) Find a 2-fold covering map $f : X \to Y$. See if you can define such a covering f by thinking about X and Y as subspace of $\mathbb{R}^3$. See if you can define such a covering by thinking about X and Y as polygonal presentations.

(b) Calculate the Euler characteristics of X and Y in this example. Show that the Euler characteristic of X is 2 times the Euler characteristic of Y.

(5) Generalize your previous example: Keep Y as the genus 2 surface, but now find another surface X_d and a d-fold covering map $f : X_d \to Y$. Try to describe f from both points of view (the surfaces as subspaces of $\mathbb{R}^3$ and as polygonal representations). What is the Euler characteristic of X_d and how does it relate to the Euler characteristic of Y?

(6) Suppose $f : A \to B$ is any d-fold covering map from some (compact) surface A to another (compact) surface B. Make a conjecture relating the Euler characteristic of A with the Euler characteristic of B. Try to write down at least some intuitive ideas and/or pictures for why this conjecture might be true (or even better, you can try to write down a proof).