HOMEOMORPHISMS OF THE TORUS

MATH 180, WINTER 2023

As you research, you may find more examples, definitions, and questions, which you definitely should feel free to include in your notes and presentation, but make sure you at least answer the following questions.

First we will think about homeomorphisms of the torus, described using matrices. In order to understand this, we need to describe the torus in a slightly new way. Consider $\mathbb{R}^2$ with the equivalence relation $(x, y) \sim (x + n, y + m)$ for any integers $n, m \in \mathbb{Z}$. The torus T^2 can be viewed as the quotient space of $\mathbb{R}^2$ by this equivalence relation $T^2 = \mathbb{R}^2/\sim$ (this is also sometimes written as $T^2 = \mathbb{R}^2/\mathbb{Z}^2$).

(1) Explain in words how this description of the torus as $\mathbb{R}^2/\sim$ gives the same answer as the description of the torus as the square with opposite sides glued together.

(2) Now consider a 2×2 matrix with integer entries:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We can define a map $A : T^2 \to T^2$ that sends a point $[(x, y)] \in T^2 = \mathbb{R}^2/\sim$ to

$$A([(x,y)]) = \left[\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right] = [(ax + by, cx + dy)]$$

(a) Prove that A is well-defined. Namely, if $(x, y) \sim (x + n, y + m)$ prove that the definition of A gives equivalent (by $\sim$) values for $A(x, y)$ and $A(x + n, y + m)$.

(b) If the matrix A has an entry which is not an integer, prove that the map defined by A is *not* well-defined as a map from $\mathbb{R}^2/\sim$ to $\mathbb{R}^2/\sim$.

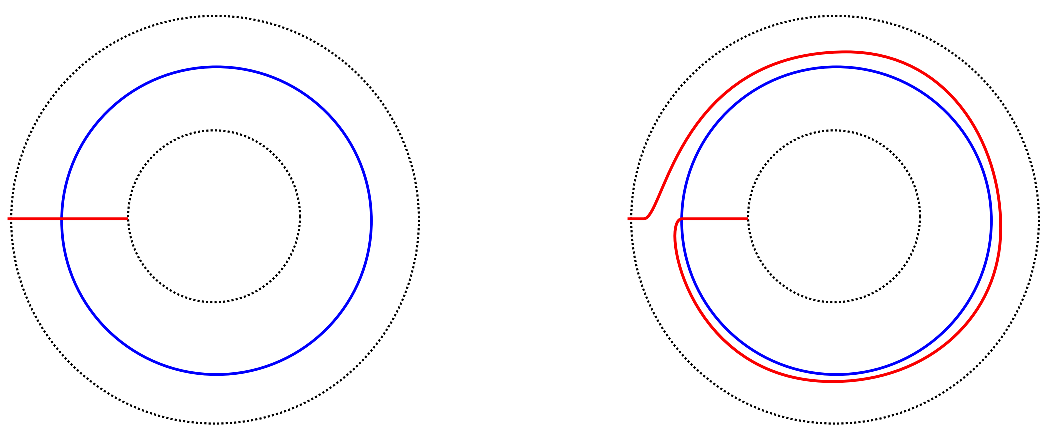
(c) Show that if the 2×2 matrix A has integer entries, then it has an inverse matrix A^{-1} with integer entries if and only if $\det(A) = 1$.

(d) Put this together to show that if A is a 2×2 matrix with integer entries and $\det(A) = 1$, then it defines a homeomorphism from T^2 to T^2.

Notice that every equivalence class in $\mathbb{R}^2/\sim$ has a representative in the square $[0, 1] \times [0, 1]$. A closed curve in the torus will look like an arc in the square (where the end points lie on points on the sides of the square which get identified to form a closed loop). In $\mathbb{R}^2$ it would look like this same arc repeated like a tile in every 1×1 square.

(3) Draw the following curves of the torus on the square (measure carefully and make sure that you align points which are identified on the left and right sides or top and bottoms sides):
   (a) $C_1 = \{(x, 0)\}$
   (b) $C_2 = \{(0, y)\}$
   (c) $C_3 = \{(2t, 3t)\}$
   (d) $C_4 = \{(t, 4t)\}$
   (e) Consider the homeomorphism defined by the matrix
$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$
   Check the determinant is 1 so this actually gives a homeomorphism. Then find the image of C_1 and C_2 under the homeomorphism A. Describe the curves in the images with equations and draw them on the square.

Another way to define homeomorphisms on surfaces is using *Dehn twists*. A Dehn twist is specified by a curve. If we look at an open neighborhood of a curve, it looks like an annulus as below. The Dehn twist is defined by cutting along the center curve, then twisting the center curve on one side of the cut by 360° and then gluing back together. Because after a 360° rotation, the points along the cut end up back where they started, this is a continuous map. Doing the same thing but rotating 360° in the other direction gives the inverse map, so this is a homeomorphism. Notice that this homeomorphism moves points that are near the curve where the cut and twist happened. For example, the image of the red arc under the Dehn twist is shown below. (The arc gets twisted around the curve.)



If the twisting is done as in this figure, we will call it a *left handed Dehn twist*. Twisting the opposite direction is a *right handed Dehn twist*. (These are inverses of each other if done along the same curve.)

By inserting the annulus as a subset of another surface Σ where the central curve is matched up with a chosen curve C in Σ, we can define the *Dehn twist about* C as a homeomorphism from Σ to Σ.

(4) Let $\Sigma = T^2 = \mathbb{R}^2/\sim$ as above and let $C = \{(t, \frac{1}{2})\}$. Let $\Phi : T^2 \to T^2$ be the homeomorphism given by the left handed Dehn twist about C. Draw the images of C_1, C_2, C_3, and C_4 under Φ using the square model of the torus. (You'll need to be particularly careful to line up points which are identified on the left and right and top and bottom.)

An isotopy is a 1-parameter continuous deformation. Things which are related by an isotopy are called isotopic. For example, an isotopy between two curves C_0 and C_1 is a 1-parameter family of curves C_t for $t \in [0, 1]$ which vary continuously with t. Similarly an isotopy of homeomorphisms $\Phi_0 : A \to B$ and $\Phi_1 : A \to B$ is a 1-parameter family of homeomorphisms $\Phi_t : A \to B$ varying continuously with t. We often think of two homeomorphisms as "the same" if they are isotopic, so we are interested in determining when homeomorphisms are isotopic and when they are not.

An important theorem which is useful for classifying homeomorphisms of surfaces up to isotopy is called *Alexander's Lemma* which is as follows:

Theorem 1 (Alexander's Lemma). *Suppose we have a set of closed curves $C_1, \dots, C_n \subset \Sigma$ which cuts the surface Σ into something homeomorphic to a disk. Let $\Phi_1 : \Sigma \to \Sigma$ and $\Phi_2 : \Sigma \to \Sigma$ be two homeomorphisms. Then Φ_1 is an isotopic homeomorphism to Φ_2 if and only if $\Phi_1(C_i)$ is isotopic to $\Phi_2(C_i)$ for all $i = 1, \dots, n$.*

Note that if we have a polygonal representation of the surface Σ, in Alexander's Lemma we can take all the curves given by joining the edges of the polygonal representation as $C_1, \dots, C_n$ since the polygon is homeomorphic to the disk. For example, in the torus, we can take the vertical and horizontal edges which are the curves C_1 and C_2 from problem 3.

(5) Let $\Phi_1 : T^2 \to T^2$ be the left handed Dehn twist about the curve C as in problem 4. Let $\Phi_2 : T^2 \to T^2$ be the homeomorphism of the torus determined by the matrix

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Show (using pictures) that $\Phi_1(C_1)$ and $\Phi_2(C_1)$ are isotopic and $\Phi_1(C_2)$ and $\Phi_2(C_2)$ are isotopic. Use Alexander's Lemma to conclude that Φ_1 and Φ_2 are isotopic homeomorphisms.

(6) Follow the same strategy to show that the right handed Dehn twist about C is isotopic to the homeomorphism determined by the matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(7) Now consider the homeomorphism given by applying the left handed Dehn twist about the curve C two times. Find the images of C_1 and C_2 after applying the left handed Dehn twist about C twice. Compare these to the images of C_1 and C_2 under the homeomorphism given by the matrix

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

Show by Alexander's Lemma that these two homeomorphisms are isotopic. How else would you know that these two homeomorphisms are the same (isotopic)? [Hint: the compositions of isotopic homeomorphisms are isotopic.]

(8) Consider another curve $D = \{(\frac{1}{2}, t)\}$ in the torus. Show that a left handed Dehn twist around D is isotopic to the homeomorphism determined by the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(9) Find the inverse matrix of

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Why is this inverse matrix isotopic to the right handed Dehn twist around D?

(10) Show using matrix multiplication that you can write any matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $\det(A) = 1$ as a product of copies of the matrices

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Using this, explain how you can get (an isotopic version of) the homeomorphism of the torus

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

as a repeated product of left/right handed Dehn twists about the curves C and/or D.