MORE HOMEOMORPHISMS OF SURFACES

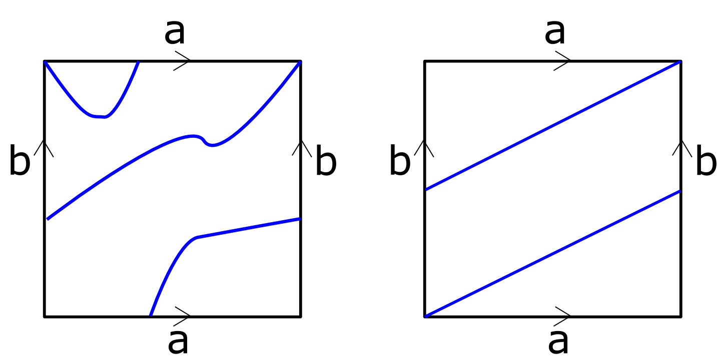
MATH 180, WINTER 2023

As you research, you may find more examples, definitions, and questions, which you definitely should feel free to include in your notes and presentation, but make sure you at least answer the following questions.

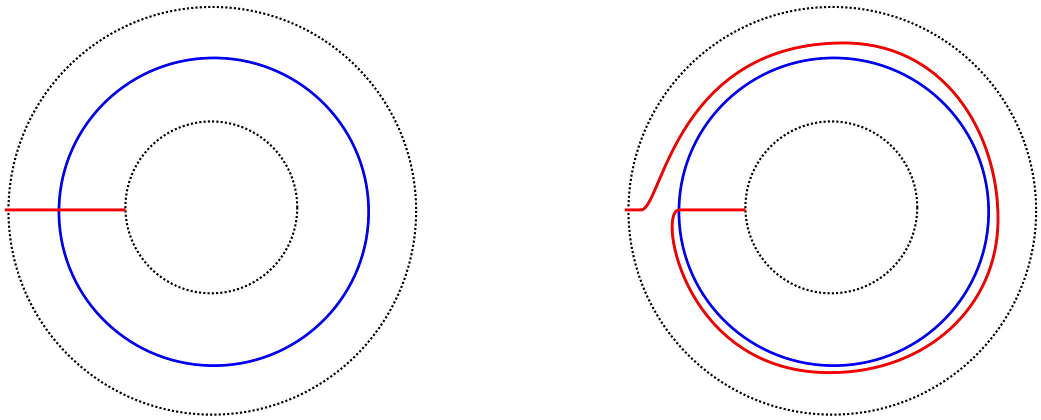
An isotopy is a 1-parameter continuous deformation. Things which are related by an isotopy are called isotopic. For example, an isotopy between two curves C_0 and C_1 is a 1-parameter family of curves C_t for $t \in [0, 1]$ which vary continuously with t.

If we think of the torus using the polygonal representation of the square with opposite sides glued, we can draw curves as arcs in the square where the end points of the arcs get glued together. You can push pieces of arc through the identifications from one side to the other, but you need to be careful that as you push, you always have endpoints matching up on the top and bottom and left and right (so you aren't cutting open the curve).

(1) Show that the curves in the below pictures on the torus are isotopic by drawing pictures and explaining what the 1-parameter family connecting the two curves does.



One way to define homeomorphisms on surfaces is using *Dehn twists*. A Dehn twist is specified by a curve. If we look at an open neighborhood of a curve, it looks like an annulus as below. The Dehn twist is defined by cutting along the center curve, then twisting the center curve on one side of the cut by 360° and then gluing back together. Because after a 360° rotation, the points along the cut end up back where they started, this is a continuous map. Doing the same thing but rotating 360° in the other direction gives the inverse map, so this is a homeomorphism. Notice that this homeomorphism moves points that are near the curve where the cut and twist happened. For example, the image of the red arc under the Dehn twist is shown below. (The arc gets twisted around the curve.)



If the twisting is done as in this figure, we will call it a *left handed Dehn twist*. Twisting the opposite direction is a *right handed Dehn twist*. (These are inverses of each other if done along the same curve.)

By inserting the annulus as a subset of another surface Σ where the central curve is matched up with a chosen curve C in Σ, we can define the *Dehn twist about* C as a homeomorphism from Σ to Σ.

(2) Let Σ be the torus, thought of using the square polygonal representation. Let $C = \{(t, \frac{1}{2})\}$ be the horizontal curve passing through the middle of the square and let $D = \{(\frac{1}{2}, t)\}$ be the vertical curve passing through the middle of the square. (Note that the start and endpoints of C get glued together so C is a closed loop, similarly with D.) Let $\Phi_C : T^2 \to T^2$ be the homeomorphism given by the left handed Dehn twist about C.
   (a) Find the image of D under Φ_C.
   (b) Find the image of D under Φ_C^{-1}.
   (c) Find the image of C under Φ_D.
   (d) Find the image of C under Φ_D^{-1}.
   (e) Show using a picture that of the four resulting curves in the above parts, two are isotopic to each other, and the other two are isotopic to each other (which pairs?).

Similarly an isotopy of homeomorphisms $\Phi_0 : A \to B$ and $\Phi_1 : A \to B$ is a 1-parameter family of homeomorphisms $\Phi_t : A \to B$ varying continuously with t. We often think of two homeomorphisms as "the same" if they are isotopic, so we are interested in determining when homeomorphisms are isotopic and when they are not.

An important theorem which is useful for classifying homeomorphisms of surfaces up to isotopy is called *Alexander's Lemma* which is as follows:

Theorem 1 (Alexander's Lemma). *Suppose we have a set of closed curves $C_1, \ldots, C_n \subset \Sigma$ which cuts the surface Σ into something homeomorphic to a disk. Let $\Phi_1 : \Sigma \to \Sigma$ and*

$\Phi_2 : \Sigma \to \Sigma$ be two homeomorphisms. Then Φ_1 is an isotopic homeomorphism to Φ_2 if and only if $\Phi_1(C_i)$ is isotopic to $\Phi_2(C_i)$ for all $i = 1, \ldots, n$.

Note that if we have a polygonal representation of the surface Σ, in Alexander's Lemma we can take all the curves given by joining the edges of the polygonal representation as $C_1, \ldots, C_n$ since the polygon is homeomorphic to the disk. For example, in the torus, we can take the vertical and horizontal edges as C_1 and C_2 (note each curve appears as two edges but they are identified by the gluing).

(3) Let C and D be the curves specified above in problem 2, and C_1 and C_2 be the horizontal and vertical edges of the square. Let Φ_C be the left handed Dehn twist around C, and Φ_D be the left handed Dehn twist around D.
   (a) Determine what is the curve $E = \Phi_C(D)$. Isotope it to "straighten it out".
   (b) Let Φ_E denote the left handed Dehn twist around the curve E you just produced. Find the curves $\Phi_E(C_1)$ and $\Phi_E(C_2)$. Isotope them to "straighten them out" as much as you can.
   (c) Consider the composition of functions $\Phi_C \circ \Phi_D \circ \Phi_C^{-1}$ (i.e. first do a right handed Dehn twist around C, then do a left handed Dehn twist around D, then do a left handed Dehn twist around C). Look at the images of C_1 and C_2 under this composition. Show that these images of C_1 and C_2 under this composition are isotopic to the images of C_1 and C_2 under Φ_E. Conclude by Alexander's Lemma that $\Phi_C \circ \Phi_D \circ \Phi_C^{-1}$ is isotopic to Φ_E.
   (d) See if you can more generally prove that for *any* curves C and D in *any* surface, doing a left handed Dehn twist around the curve $\Phi_C(D)$ is isotopic to doing the composition $\Phi_C \circ \Phi_D \circ \Phi_C^{-1}$.

(4) Using the same notation as above, consider the composition

$$\Psi = \Phi_C \circ \Phi_D \circ \Phi_C \circ \Phi_D \circ \Phi_C \circ \Phi_D.$$

Show that $\Psi(C_1)$ is isotopic to C_1 and $\Psi(C_2)$ is isotopic to C_2 (you'll need to apply the Dehn twists in order carefully, and you can always isotope the curves to simplify/straighten them out as you go) Conclude using Alexander's Lemma that Ψ is isotopic to the identity map on the torus.