BUILDING MANIFOLDS ABSTRACTLY, TRANSITION FUNCTIONS, SMOOTH STRUCTURES

MATH 180, WINTER 2023

As you research, you may find more examples, definitions, and questions, which you definitely should feel free to include, but make sure you at least go through the following discussion and questions.

Suppose X is a manifold with an atlas of coordinate charts $\{\phi_i : U_i \to V_i\}_{i\in\mathcal{I}}$.

The first goal of this project is to see that we can represent a manifold X by taking open subsets of $\mathbb{R}^n$ (the V_i), and gluing them together using gluing maps and a corresponding equivalence relation with the quotient topology.

First we define which portions of the V_i will get glued together:

For each pair $i, j \in \mathcal{I}$ where $U_i \cap U_j \neq \emptyset$, we define $V_{i,j} \subset V_i$ as $V_{i,j} = \phi_i(U_i \cap U_j)$ (and correspondingly by switching the roles of i and j we get that $V_{j,i} = \phi_j(U_j \cap U_i) \subset V_j$).

Next we define the gluing maps:

The *transition function* from $V_{i,j}$ to $V_{j,i}$ is defined by $\phi_{i,j} : V_{i,j} \to V_{j,i}$ $\phi_{i,j}(x) = \phi_j \circ \phi_i^{-1}(x)$.

Treating the V_i as open subsets in different copies of Euclidean space (so they are initially considered disjoint before gluing), we can define the following quotient space:

$$Y = \left(\bigsqcup_{i\in\mathcal{I}} V_i\right) /\!\sim \qquad\qquad x \sim \phi_{i,j}(x) \text{ for } x \in V_{i,j}, \text{ for } i, j \in \mathcal{I}$$

Let $[y]$ denote the equivalence class of a point for $y \in \sqcup_{i\in\mathcal{I}} V_i$ under the equivalence relation given by the transition functions.

Denote the quotient map by the equivalence relation by

$$q : \bigsqcup_{i\in\mathcal{I}} V_i \to Y$$

defined by $q(y) = [y]$, and use the quotient topology to define open subsets in the quotient space ($U \subset Y$ is open if and only if $q^{-1}(U)$ is open).

(1) Consider the map $F : X \to Y$ defined by $F(x) = [\phi_i(x)]$ for $x \in U_i$.
   (a) Prove that F is well defined: if x is in U_i and U_j then $F(x) = [\phi_i(x)]$ and $F(x) = [\phi_j(x)]$ so prove that in this situation $[\phi_i(x)] = [\phi_j(x)]$ (i.e. show that $\phi_i(x) \sim \phi_j(x)$).
   (b) Prove that F is continuous.

   Now consider the map $G : Y \to X$ defined by $G([y]) = \phi_i^{-1}(y)$ if $y \in V_i$.

(c) Prove that G is well-defined: if $y' \sim y$ and $y' \in V_j$, show that $\phi_j^{-1}(y') = \phi_i^{-1}(y)$.
(d) Prove that G is continuous.
(e) Prove that F and G are inverse functions: show that $G(F(x)) = x$ for any $x \in X$ and $F(G([y])) = [y]$ for any $[y] \in Y$.

Conclude that F is a homeomorphism between X and Y, so X and Y are homeomorphic.

(2) Consider $S^1 \subset \mathbb{R}^2$ with the four coordinate charts coming from projections. Using this atlas, find all the $V_{i,j}$ and transition functions $\phi_{i,j}$. Draw a picture to demonstrate how the pieces are glued together.
(3) Choose an atlas for S^2, and write down all the coordinate charts explicitly. Then find the corresponding $V_{i,j}$ and transition functions $\phi_{i,j}$. Draw a picture to indicate how the pieces are glued together.

The only property of Euclidean space we have focused on in the course is its open subsets (using ε balls), and continuous maps and homeomorphisms between Euclidean spaces (note that homeomorphisms are exactly the bijections which take open sets to open sets). However, there are other properties and structures that one can think about on Euclidean space.

Instead of just asking maps between Euclidean spaces to be continuous, we can also ask them to be differentiable (and we can ask their partial derivatives are also differentiable and those (second) partial derivatives also be differentiable etc.). If we think about a manifold as just a bunch of open subsets of Euclidean space glued together, then each Euclidean piece comes with a notion of taking derivatives. If the gluing maps (transition functions) themselves are infinitely differentiable, the notions of taking derivatives on different V_i will be compatible with each other. This gives the manifold a *smooth structure* or *differentiable structure* (i.e. a notion of calculus on the entire manifold).

(4) For the transition functions for your atlases for S^1 and S^2 above, check that these transition functions are infinitely differentiable on their domains $V_{i,j}$. This gives S^1 and S^2 smooth structures.
(5) Search and read a bit about "Milnor's exotic 7-spheres". You can look at different sources online, but also try to take a look at Milnor's original article "On manifolds homeomorphic to the 7-sphere." You don't need to understand everything in the article, but see how much you can make sense of. What is the main result? Can you understand the outline of the proof or a key construction? Give a short summary of the ideas you can gather.