HOMEWORK 1 (MAT 215A) TOPOLOGY

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Reminder: Continuity means that the pre-image of any open set is open, and two topological spaces X, Y are homeomorphic if there exist *continuous* maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$. Use the subspace topology, product topology, and quotient topology as needed.

Problems:

(1) Prove that $\mathbb{R}\mathrm{P}^1$ is homeomorphic to S^1. (Use the quotient topology to define the topology on $\mathbb{R}\mathrm{P}^1$ and the subspace topology (and Euclidean topology) to define the topology on S^1.)

(2) Prove that $\mathbb{R}\mathrm{P}^n = A \sqcup B$ where A is homeomorphic to $\mathbb{R}\mathrm{P}^{n-1}$ and B is homeomorphic to $\mathbb{R}^n$. Use this to inductively prove that $\mathbb{R}\mathrm{P}^n = A_0 \sqcup A_1 \sqcup \cdots \sqcup A_n$ where A_i is homeomorphic to $\mathbb{R}^i$.

(3) Define complex projective space $\mathbb{C}\mathrm{P}^n$ as the space of complex lines in $\mathbb{C}^{n+1}$. Write down the corresponding homogeneous coordinate description. Then show that $\mathbb{C}\mathrm{P}^n = B_0 \sqcup B_1 \sqcup \cdots \sqcup B_n$ where B_i is homeomorphic to $\mathbb{C}^i$.

(4) Show that homotopy gives an equivalence relation on maps $f : X \to Y$. Namely check the three properties
   (a) Reflexivity: $f \simeq f$.
   (b) Symmetry: If $f \simeq g$ then $g \simeq f$.
   (c) Transitivity: If $f \simeq g$ and $g \simeq h$ then $f \simeq h$.

(5) Prove that the following two maps are homotopic: $f_0, f_1 : S^1 \to \mathbb{R}^2 \backslash \{0\}$ where $f_0(x_1, x_2) = (x_1, x_2)$, $f_1(x_1, x_2) = (4x_1, \frac{1}{3}x_2)$.

(6) Prove that for any space X, $X \times \mathbb{R}^n$ is homotopy equivalent to X.

(7) Prove that the torus with a disk deleted is homotopy equivalent to the wedge of two circles.