MIDTERM (215A) TOPOLOGY

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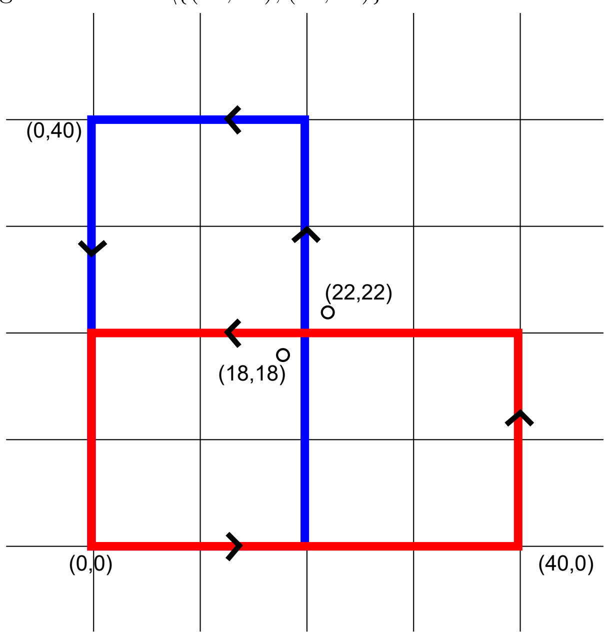
Rules:

- You may not discuss anything about the problems on this midterm with any other person, except to ask questions via email to Laura Starkston.
- You may use Hatcher and your class notes as a reference. You may not use the internet as a reference.
- You may quote theorems that were proven in class or are proven in Hatcher in your proofs.
- You may assume that continuous maps between Euclidean spaces are continuous, without proof.

(1) Write down an explicit homotopy (in coordinates) between the following two loops based at the origin in $X = \mathbb{R}^2 \setminus \{(18,18),(22,22)\}$.



$$\gamma_0(t) = \begin{cases} (160t, 0) & 0 \leqslant t \leqslant \frac{1}{4} \\ (40, 80t - 20) & \frac{1}{4} \leqslant t \leqslant \frac{1}{2} \\ (120 - 160t, 20) & \frac{1}{2} \leqslant t \leqslant \frac{3}{4} \\ (0, 80 - 80t) & \frac{3}{4} \leqslant t \leqslant 1 \end{cases}$$

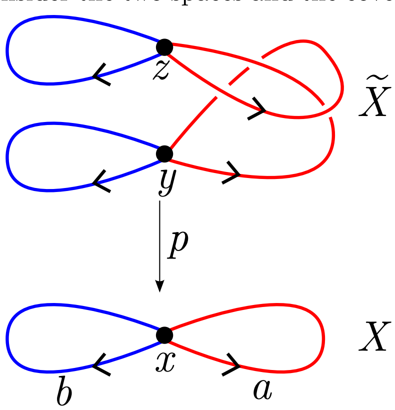
$$\gamma_1(t) = \begin{cases} (80t, 0) & 0 \leqslant t \leqslant \frac{1}{4} \\ (20, 160t - 40) & \frac{1}{4} \leqslant t \leqslant \frac{1}{2} \\ (60 - 80t, 40) & \frac{1}{2} \leqslant t \leqslant \frac{3}{4} \\ (0, 160 - 160t) & \frac{3}{4} \leqslant t \leqslant 1 \end{cases}$$

(2) Let $X = T^2 \times I/\sim$ (here $T^2 = S^1 \times S^1$ is the torus and $I = [0,1]$) where the equivalence relation $\sim$ is defined by $(p,0) \sim (f(p),1)$ for every $p \in T^2$ where $f : T^2 \to T^2$ is the homeomorphism defined by identifying $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and letting $f([(x,y)]) = [(x, nx + y)]$ where $n \in Z$. (X is called the mapping torus of f.) Let x_0 be the equivalence class of the point $([(0,0)],0)$. Give a presentation for the group $\pi_1(X, x_0)$ using Seifert-van Kampen.

(3) Here we will examine the fundamental group of $\mathbb{R}\mathrm{P}^3$.
   (a) Let $p_0 = [0:0:0:1]$. Calculate the fundamental group $\pi_1(\mathbb{R}\mathrm{P}^3, p_0)$ using the cell decomposition and the Seifert van Kampen theorem.
   (b) Show that the map $q : S^3 \to \mathbb{R}\mathrm{P}^3$ obtained by the quotient relation $q(x_0, x_1, x_2, x_3) = [x_0 : x_1 : x_2 : x_3]$ is a covering map.
   (c) What is $q^{-1}([0:0:0:1])$?
   (d) Let $s_0 = (0,0,0,1)$. What is the covering subgroup $q_*(\pi_1(S^3, s_0)) \subseteq \pi_1(\mathbb{R}\mathrm{P}^3, p_0)$? Check that the quotient is in bijection with $q^{-1}([0:0:0:1])$.
   (e) Without using Seifert van Kampen, and only using covering spaces, how could you have calculated $\pi_1(\mathbb{R}\mathrm{P}^3, p_0)$?

(4) Let $X = S^1_1 \vee S^1_2$ where S^1_1 and S^1_2 are both copies of the circle. Let $A \subset X$ be the subset S^1_1 included into the wedge sum. Construct an infinite family of retractions $r_n : S^1_1 \vee S^1_2 \to S^1_1$ such that if $n \neq m$ r_n is not homotopic to r_m. (Define r_n and prove that the maps are not homotopic).

(5) Consider the two spaces and the covering map between them indicated by the projection $p : \widetilde{X} \to X$.



   (a) Give a presentation for $\pi_1(\widetilde{X}, y)$. Draw the generating loops on a copy of the picture of $\widetilde{X}$.
   (b) Given generators a and b as shown for $\pi_1(X, x)$, determine where p_* sends each of your generators of $\pi_1(\widetilde{X}, y)$. Use this to give a presentation for the subgroup $p_*(\pi_1(\widetilde{X}, y))$.
   (c) Show explicitly in this example the bijection between the cosets $p_*(\pi_1(\widetilde{X}, y))\backslash\pi_1(X, x)$ with the points of $p^{-1}(x)$.
   (d) Define a space Y and a covering map $p : Y \to X$ such that the covering group $p_*(\pi_1(Y, y_0))$ is the subgroup of $\pi_1(X, x)$ generated by $a^3, b, aba^{-1}, a^2ba^{-2}$.
   (e) Prove that for every $n \geqslant 2$, there exists a space $\widetilde{X}_n$ such that $\widetilde{X}_n$ is homotopy equivalent to the wedge of n circles and there exists a covering map $p : \widetilde{X}_n \to X$. ($X$ is still the wedge of two circles as above.)

(6) Consider the maps $p_k : S^1 \to S^1$ defined by $p_k(e^{i\theta}) = e^{ik\theta}$. Let $1 = e^{i\cdot 0}$ be a base point in S^1. Recall that each of these maps is a covering map when k is a positive integer.

(a) What is the covering group $(p_k)_*(\pi_1(S^1, 1)) \subset \pi_1(S^1, 1) \cong \mathbb{Z}$?

(b) What if p_k and p_ℓ are two such covering maps, what condition on the positive integers k and ℓ ensures that $(p_k)_*(\pi_1(S^1, 1)) \subset (p_\ell)_*(\pi_1(S^1, 1))$?

(c) Find explicitly a lift g (define $g : S^1 \to S^1$) such that the following diagram commutes, assuming that your criterion in terms of k and ℓ in the previous part is satisfied.

$$\begin{tikzcd} & S^1 \arrow[d, "p_\ell"] \\ S^1 \arrow[ur, dotted, "g"] \arrow[r, "p_k"] & S^1 \end{tikzcd}$$