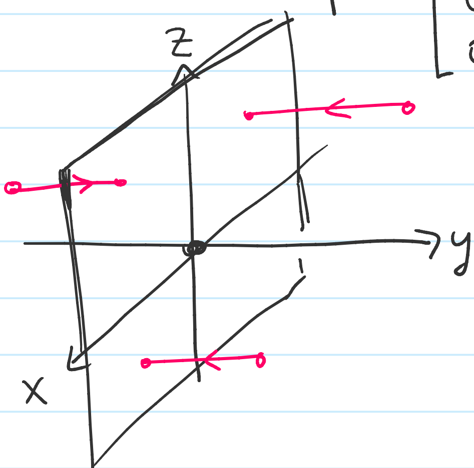
MAT 022A Nov 4, 2019

Thursday, October 31, 2019 1:53 PM

Projection Matrices

Example 1:

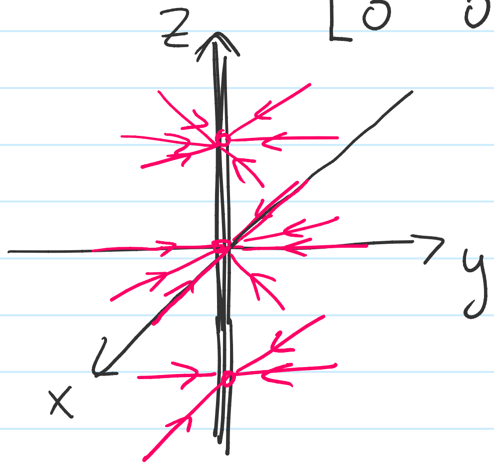
$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}$$



P projects vectors in $\mathbb{R}^3$ onto the xz plane

Example 2:

$$P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Px = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$$



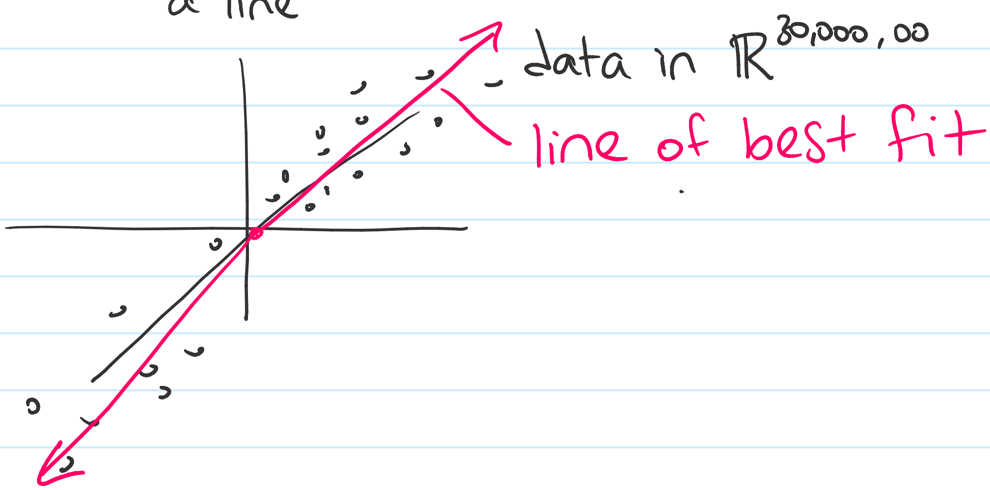
P projects vectors in $\mathbb{R}^3$ onto the z-axis.

Question: How do we project onto other planes, lines, subspaces?

<u>Motivation</u>: Data with 30,000,000 parameters (in $\mathbb{R}^{30{,}000{,}000}$) takes up a lot of space

Projecting it to a smaller dimensional subspace makes it take up less space.

First we will study how to project down to a $\underbrace{\text{1-dimensional subspace}}_{\text{a line}}$ (1 parameter)



A line has one vector in its basis.

<u>Example</u>: Suppose we want to project $\mathbb{R}^5$ to the line

$$L = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 3 \end{bmatrix} \right)$$

$$V = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}$$

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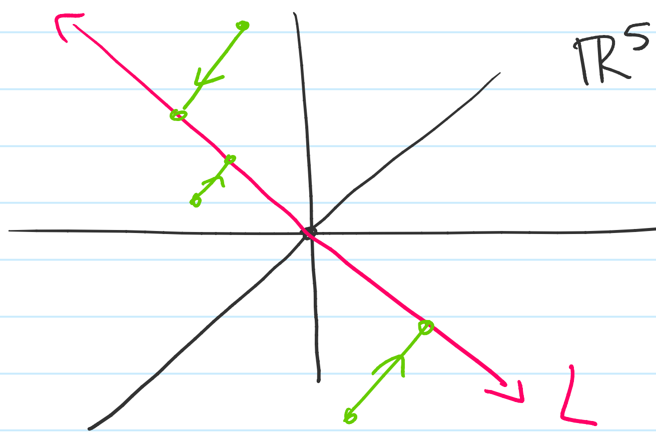
3 Key properties of the projection matrix P:

① For every vector $\vec{x}$ in $\mathbb{R}^5$ $P\vec{x}$ should be in the line L.

$\rightsquigarrow$ $P\vec{x}$ must be in $\text{Span}(v) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}\right)$



What do we know about column spaces?

$P\vec{x}$ is always in $C(P)$.

<u>Conclusion 1</u>: Want $C(P) = \text{Span}(v) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}\right)$

So the columns of P should be scalar multiples of v.

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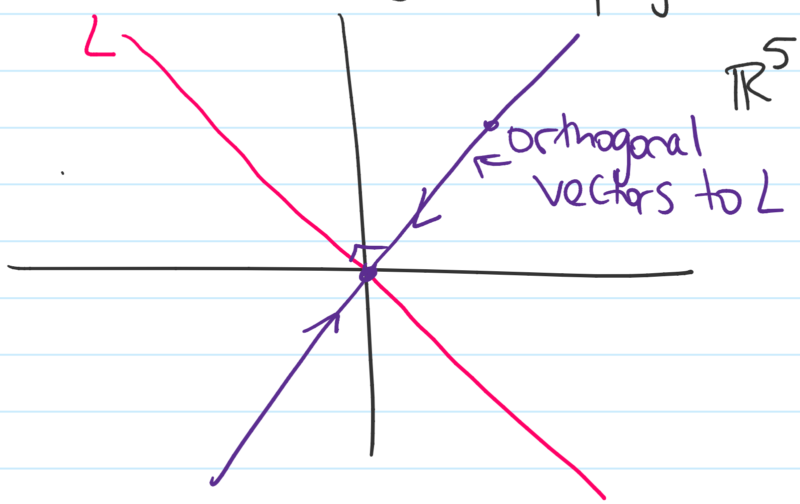
(2) Every vector in $\mathbb{R}^5$ which is orthogonal to the line L should project to $\vec{0}$



In other words, if $\vec{x}$ is orthogonal to L, $P\vec{x} = 0$.

So the subspace orthogonal to L should be the nullspace $N(P)$.

<u>Remember</u>: $N(P)$ is orthogonal to $C(P^T)$

So if $C(P^T) = L$, $N(P)$ will be orthogonal to L.

Therefore we want the columns of P^T to be scalar multiples of V

Columns of $P^T \longrightarrow$ Rows of P

<u>Conclusion 2</u>: We want the rows of P to be multiples of $V^T = [1 \;\; 2 \;\; -1 \;\; 0 \;\; 3]$

A matrix satisfying conclusions 1 and 2 has the form

$$c\,v\,v^T = c\begin{bmatrix}1\\2\\-1\\0\\3\end{bmatrix}\begin{bmatrix}1 & 2 & -1 & 0 & 3\end{bmatrix}$$

$$= c\begin{bmatrix}1 & 2 & -1 & 0 & 3\\2 & 4 & -2 & 0 & 6\\-1 & -2 & 1 & 0 & -3\\0 & 0 & 0 & 0 & 0\\3 & 6 & -3 & 0 & 9\end{bmatrix}$$

Rows and columns are multiples of v

Need one last key property:

(3) $Pv = v$ (a vector in the line projects to itself)

$$\underbrace{c\begin{bmatrix}1\\2\\-1\\0\\3\end{bmatrix}\begin{bmatrix}1 & 2 & -1 & 0 & 3\end{bmatrix}}_{P}\underbrace{\begin{bmatrix}1\\2\\-1\\0\\3\end{bmatrix}}_{v} = c\underbrace{\begin{bmatrix}1\\2\\-1\\0\\3\end{bmatrix}}_{v}\underbrace{\begin{bmatrix}1^2+2^2+(-1)^2+0^2+3^2\end{bmatrix}}_{v\cdot v}$$

If we want $Pv = v$, make $c = \frac{1}{v\cdot v} = \frac{1}{v^Tv}$

<u>Projection formula</u>: $P = \frac{1}{v^Tv}\cdot v\cdot v^T$

Example: Find the projection of the vector

$\vec{b} = \begin{bmatrix} 2 \\ 2 \\ -2 \\ 1 \\ 2 \end{bmatrix}$ onto the line $L = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 3 \end{bmatrix} \right)$

Projection matrix $P = \frac{1}{v^T v} \cdot v v^T$

$$= \frac{1}{1+4+1+0+9} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \end{bmatrix}$$

Projection of $\vec{b}$ to the line L is

$$P\vec{b} = \frac{1}{15} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ -2 \\ 1 \\ 2 \end{bmatrix}$$

$$= \left(\frac{1}{15}\right) \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 3 \end{bmatrix} \begin{bmatrix} 2+4+2+0+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} \frac{14}{15} \end{bmatrix} = \boxed{14/15 \, v} = \begin{bmatrix} 14/15 \\ 28/15 \\ -14/15 \\ 0 \\ 42/15 \end{bmatrix}$$

General formula to project to a subspace U.

① Find a basis for U

Example $U = \text{Span}\left(\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right)$

linearly independent

② Create a matrix with the basis vectors as columns

$$A = \begin{bmatrix}1 & 0\\2 & 1\\0 & -1\end{bmatrix}$$

③ Projection matrix formula:

$$\boxed{P = A(A^TA)^{-1}A^T}$$

(This generalizes the line case when $P = \frac{vv^T}{v^Tv} = v(v^Tv)^{-1}v^T$)

In this example:

$$P = \begin{bmatrix}1 & 0\\2 & 1\\0 & -1\end{bmatrix}\left(\begin{bmatrix}1 & 2 & 0\\0 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\2 & 1\\0 & -1\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 2 & 0\\0 & 1 & -1\end{bmatrix}$$

Need to multiply
then calculate inverse