$f: X \to Y$ $\quad$ X, Y smooth manifolds $\quad$ f smooth $\xrightarrow{\text{defn}}$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi"] \arrow[r, "g"] & V \arrow[u, "\psi"'] \end{tikzcd} \qquad U \subset \mathbb{R}^n, \quad V \subset \mathbb{R}^m$$

arbitrary choices of parametrizations

$g := \psi^{-1} \circ f \circ \varphi$

Say f is smooth if g is smooth in the usual analysis sense

$f: X \to Y$ $\quad$ open subset of $\mathbb{R}^{\cdot}$ $\to$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi"] \arrow[r, "\text{std form}"'] & V \arrow[u, "\psi"'] \end{tikzcd}$$

	defn (linear algebra of df_x)	$\exists$ coords s.t. f has standard form	Preimage theorems	
immersion	df_x injective $\forall x \in X$	$i: \mathbb{R}^k \to \mathbb{R}^n$ $k \leq n$ $(x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, 0, \dots, 0)$		
submersion	df_x surjective $\forall x \in X$	$p: \mathbb{R}^m \to \mathbb{R}^n$ $m \geq n$ $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_n)$	If y is a regular value $f^{-1}(y)$ is a submanifold of X	$\dim(f^{-1}(y)) = \dim X - \dim Y$
local diffeo	df_x isomorphism $\forall x \in X$	identity		
transverse to $Z \subset Y$ ($\uparrow$ submanifold)	$\text{image}(df_x) + T_{f(x)}Z = T_{f(x)}Y$ $\forall x \in f^{-1}(Z)$	$Z = (x_1, \dots, x_k, 0, \dots, 0)$ $t: \mathbb{R}^m \to \mathbb{R}^n$ $(x_1, \dots, x_m) \mapsto (0, \dots, 0, x_1, \dots, x_\ell)$ $n - \ell \leq k$	$f \pitchfork Z$ $f^{-1}(Z)$ is a submfld of X	$\dim(f^{-1}(Z)) = \dim X - \dim Y + \dim Z$

To prove local forms:

$$\begin{tikzcd} x & X \arrow[r, "f"] & Y & f(x) \\ 0 \arrow[u, mapsto] & U \arrow[u, "\varphi"] \arrow[r, "g"] & V \arrow[u, "\psi"'] & 0 \arrow[u, mapsto] \end{tikzcd}$$

$g = \psi^{-1} \circ f \circ \varphi$ $\quad$ Goal to change φ, ψ until g is standard model

- Look at dg_0, the defining property + linear algebra $\Rightarrow$ $\exists$ change of basis to transform dg_0 into a matrix that looks like $d(\text{standard model})_0$

Euclidean (matrix) version of df_x

Use change of basis transformations, thinking of them as smooth maps $\mathbb{R}^{\cdot} \to \mathbb{R}^{\cdot}$, to modify φ, ψ by composing w/ change of basis

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi"] \arrow[r, "g"] & V \arrow[u, "\psi"'] \\ \tilde{U} \arrow[u, "A"] \arrow[r, "\tilde{g}"] & \tilde{V} \arrow[u, "B"'] \end{tikzcd}$$

change of basis $\to$ A $\quad$ B $\leftarrow$ change of basis $\quad$ new g

Now with new g ($\tilde{g}$), $dg_0 = d(\text{standard model})_0$

Want to use inverse function theorem

Find G defined between open subsets of $\mathbb{R}^{\cdot}$ such that

- G composes (in some order) with std model to equal g.
- dG_0 is an isomorphism

Then use inverse fct thm to say G is a local diffeo,

compose either φ or ψ with G to get better φ or ψ

- because G local diffeo, still get parametrization after composing φ/ψ with G/G^{-1}
- because G composed w/ std model to give g

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi"] \arrow[r, "g"] & V \arrow[u, "\psi"'] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "f"] & & Y \\ U \arrow[u, "\varphi"] \arrow[r, "G"] & W \arrow[r, "\text{std}"] & V \arrow[u, "\psi"'] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] & Y \\ W \arrow[u, "\varphi"] \arrow[r, "\text{std}"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

depending

Preimage theorems:

Regular value case: $f^{-1}(y)$ Standard model for submersion, preimage of a point

$m \geq n \quad p: (x_1, \dots, x_m) \mapsto (x_1, \dots, x_n)$

$$\begin{aligned} p^{-1}(0, \dots, 0) &= \{(0, \dots, 0, x_{n+1}, \dots, x_m) \mid x_{n+1}, \dots, x_m \in \mathbb{R}\} \\ &= 0 \times \mathbb{R}^{m-n} \\ &\cong \mathbb{R}^{m-n} \\ & \mathbb{R}^{\dim X - \dim Y} \end{aligned}$$

Transverse case: Reduced to regular value case

Locally represent $Z \subset Y$ as $(x_1, \dots, x_k, 0, \dots, 0)$

$$\begin{array}{c} Y \\ \cup \\ W \\ \psi \uparrow \downarrow \psi^{-1} \\ V \\ \cap \\ \mathbb{R}^n \end{array} \qquad \begin{aligned} &\psi^{-1}: W \to V, \quad W \subset Y, \ V \subset \mathbb{R}^n \\ &\psi^{-1}(y) = (g_1(y), \dots, g_n(y)) \\ &Z \cap W = \{ g_{k+1}(y) = \dots = g_n(y) = 0 \mid y \in W \subset Y \} \end{aligned}$$

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & & \\ \cup & & \cup & & \\ f^{-1}(W) & \xrightarrow{f} & W & \xrightarrow{g} & \mathbb{R}^{n-k} \end{array} \qquad g(y) = (g_{k+1}(y), \dots, g_n(y))$$

Transverse condition for f, + fact that dg_w is surjective $\Rightarrow$ $g \circ f$ submersion at $x \in f^{-1}(Z) \cap W$

Use regular value preimage theorem.