MAT 239: HOMEWORK 1

(1) **Stereographic projection:** Stereographic projection defines a diffeomorphism $p_N : S^2 \setminus \{N\} \to \mathbb{R}^2$ where $N = (0, 0, 1)$ is the north pole as follows. Embed $\mathbb{R}^2$ in $\mathbb{R}^3$ as the points where $z = 0$. Then for any point $(x_0, y_0, z_0) \in S^2 \setminus \{N\}$, the line through (x_0, y_0, z_0) intersects $\mathbb{R}^2$ at a unique point $p_N((x_0, y_0, z_0))$.

  (a) Find an explicit formula for $p_N((x_0, y_0, z_0))$.

  (b) Find an explicit formula for $p_N^{-1} : \mathbb{R}^2 \to S^2 \setminus \{N\}$.

  (c) Define an analogous projection $p_S : S^2 \setminus \{S\} \to \mathbb{R}^2$, and compute the transition function $p_S \circ p_N^{-1}$ on the domain where they are commonly defined.

  (d) Generalize stereographic projection to define a diffeomorphism $p_N^k : S^k \setminus \{N\} \to \mathbb{R}^k$.

(2) **Tangent space to the sphere:** Consider the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

  (a) Determine the tangent space $T_{(x,y,z)}S^2$.

  (b) Let $f : S^2 \to \mathbb{R}$ be the restriction of the projection $f(x, y, z) = x$. Calculate

$$df_{(x,y,z)} : T_{(x,y,z)}S^2 \to T_x\mathbb{R}$$

  and determine at which points (x, y, z) is this the zero map.

(3) **The 2-Torus:** The 2-torus T^2 is defined to be $S^1 \times S^1$. We can realize this as a subset of $\mathbb{R}^N$ in different ways.

  (a) Viewing $S^1 \subset \mathbb{R}^2$, defines $T^2 = S^1 \times S^1 \subset \mathbb{R}^4$. Define diffeomorphisms from enough open subsets of $\mathbb{R}^2$ to open subsets of $T^2 \subset \mathbb{R}^4$ to cover T^2 to prove that it is a 2-dimensional manifold.

  (b) Consider the subset of points $T_{a,b}$ in $\mathbb{R}^3$ at distance b from the circle of radius a where $0 < b < a$. Prove that T^2 is diffeomorphic to this subset.

  (c) Determine the tangent space to $T_{a,b}$ at each point, presented as the span of two vectors.

(4) **Manifolds with boundary:**

  (a) Prove that the unit ball

$$B^n = \{(x_1, \dots, x_n) \mid x_1^2 + \cdots + x_n^2 \leq 1\}$$

  is a smooth n-manifold with boundary.

  (b) Prove that the unit square

$$R = \{(x, y) \mid 0 \leq x, y \leq 1\}$$

  is NOT a 2-manifold with boundary.

(5) **Projective spaces:**

(a) The *real n-dimensional projective space*, $\mathbb{RP}^n$ is the quotient of $\mathbb{R}^{n+1} \setminus 0$ by the equivalence relation $(x_1, \cdots, x_{n+1}) \sim (\lambda x_1, \cdots, \lambda x_{n+1})$ for $\lambda \in \mathbb{R} \setminus 0$. A typical way to represent $\mathbb{R}P^n$ is to use *homogeneous coordinates* as follows:

$$\mathbb{RP}^n = \{[x_1 : x_2 : \cdots : x_n, x_{n+1}] \mid (x_1, x_2, \cdots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \setminus (0, 0, \cdots, 0, 0)\}$$

where $[x_1 : x_2 : \cdots : x_n : x_{n+1}] = [\lambda x_1 : \lambda x_2 : \cdots : \lambda x_n : \lambda x_{n+1}]$.

Prove that $\mathbb{RP}^n$ is a manifold. Hint: show that the subsets of points with a representative where $x_i = 1$ give a covering by coordinate charts.

(b) The complex projective spaces, $\mathbb{CP}^n$, is defined similarly as the quotient of $\mathbb{C}^{n+1} \setminus 0$ by the equivalence relation $(z_1, \cdots, z_{n+1}) \sim (\lambda z_1, \cdots, \lambda z_{n+1})$ for $\lambda \in \mathbb{C} \setminus 0$. It also can be expressed in homogeneous coordinates $[z_1 : z_2 : \cdots : z_{n+1}] = [\lambda z_1 : \lambda z_2 : \cdots : \lambda z_{n+1}]$ for $\lambda \in \mathbb{C} \setminus 0$. Show that $\mathbb{CP}^n$ is a manifold. What is its (real) dimension?

(6) **Products:**

(a) Suppose $f : X \to X'$ and $g : Y \to Y'$ are smooth maps. Define the product map $f \times g : X \times Y \to X' \times Y'$ by

$$(f \times g)(x, y) = (f(x), g(y)).$$

Prove that $f \times g$ is smooth and that $d(f \times g)_{(x,y)} = df_x \times dg_y$.

(b) Show that for any smooth manifolds X and Y,

$$T_{(x,y)}(X \times Y) = T_x X \times T_x Y.$$

(c) Show that the projection map $\pi : X \times Y \to X$ given by $\pi(x, y) = x$ is smooth and $d\pi_{(x,y)}(v, w) = v$ for $(v, w) \in T_x X \times T_y Y = T_{(x,y)}(X \times Y)$.

(7) **Cut-off and bump functions:**

(a) Consider the following function $\rho : \mathbb{R} \to \mathbb{R}$ defined by

$$\rho(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Prove that ρ is smooth.

(b) For $a < b$, define $\sigma_{a,b}(x) = \rho(x - a)\rho(b - x)$. Prove that $\sigma_{a,b}$ is a smooth function which is positive on (a, b) and 0 elsewhere.

(c) Define

$$\tau_{a,b}(x) = \frac{\int_x^b \sigma_{a,b}(u)du}{\int_a^b \sigma_{a,b}(u)du}.$$

Prove that $\tau_{a,b}$ is a smooth function such that $\tau_{a,b}(x) = 1$ for $x \leq a$, $0 < \tau_{a,b}(x) < 1$ for $a < x < b$, and $\tau_{a,b}(x) = 0$ for $t \geq b$.

(d) Construct a smooth function on $\mathbb{R}^k$ that equals 1 on points in the ball of radius $\varepsilon > 0$, 0 outside the ball of radius $\alpha > \varepsilon$, and is strictly between 0 and 1 in between.