MAT 239: HOMEWORK 2

(1) **Chain Rule:** Assuming the chain rule for maps between open subsets of $\mathbb{R}^n$

$$f : \mathbb{R}^n \to \mathbb{R}^m,\ g : \mathbb{R}^m \to \mathbb{R}^k,\ d(g \circ f)_x = dg_{f(x)} \circ df_x,$$

prove the chain rule for maps between smooth manifolds, i.e. if X, Y, Z are smooth manifolds of dimension m, n, k respectively and $f : X \to Y$ and $g : Y \to Z$ are smooth maps, prove that

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

using parametrizations.

(2) **Vector fields on S^2:** Let

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

Prove that $\zeta : S^2 \to TS^2$ defined by

$$\zeta((x, y, z)) = ((x, y, z), \langle z, z, -x - y\rangle)$$

is a smooth vector field on S^2.

(3) TT^2 **is a trivial bundle:** Let $T^2 = S^1 \times S^1$ (you may embed in $\mathbb{R}^3$ or $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$ as in hw1).

(a) Find two vector fields ζ_1 and ζ_2 on T^2, such that for each $x \in T^2$ $(\zeta_1(x), \zeta_2(x))$ is a basis for T_xT^2. (Define ζ_1 and ζ_2, check they are smooth vector fields, and check they give a basis at each point.)

(b) Show that the map

$$\Psi : T^2 \times \mathbb{R}^2 \to TT^2$$

defined by

$$\Psi(x, (a, b)) = (x, a\zeta_1(x) + b\zeta_2(x))$$

is a diffeomorphism.

When TM is diffeomorphic to $M \times \mathbb{R}^n$, we say that TM is a *trivial bundle.*