MAT 239: HOMEWORK 3

(1) Given two parametrizations, $\phi_1 : U_1 \to X$ and $\phi_2 : U_2 \to X$ whose images overlap in X, we can choose an orientation on $U_1 \subset \mathbb{R}^n$ and an orientation on $U_2 \subset \mathbb{R}^n$, each of which induces an orientation on the portion of X in the image.

  (a) Suppose the images of ϕ_1 and ϕ_2 overlap in X. Show that the following are equivalent:

    (i) The orientation induced by $d\phi_1$ is equivalent to the orientation induced by $d\phi_2$ on the tangent spaces T_xX for points x in the intersection of the images of ϕ_1 and ϕ_2.

    (ii) For the transition function $t = \phi_2^{-1} \circ \phi_1$, dt_s takes the orientation on U_1 to an equivalent orientation on U_2.

    (iii) If $\det(dt_s)$ is positive, the orientation on U_1 is equivalent to the orientation on U_2 (as open subsets of $\mathbb{R}^n$), and if $\det(dt_s)$ is negative, the orientation on U_1 is inequivalent to the orientation on U_2 (as open subsets of $\mathbb{R}^n$).

  If any of these equivalent notions holds, we say that the orientations chosen on $\phi_1 : U_1 \to X$ and $\phi_2 : U_2 \to X$ are *compatible*.

  (b) Suppose $\phi_i : U_i \to X$ are parametrizations for $i = 1, 2, 3$ such that $\text{image}(\phi_1), \text{image}(\phi_2) \subset \text{image}(\phi_3)$ and $\text{image}(\phi_1) \cap \text{image}(\phi_2) \neq \emptyset$. Show that if we choose orientations for ϕ_1, ϕ_2, and ϕ_3, such that the ϕ_1 orientation is compatible with the ϕ_2 orientation and the ϕ_1 orientation is compatible with the ϕ_3 orientation, then the ϕ_2 orientation is compatible with the ϕ_3 orientation.

(2) Suppose X and Y are orientable manifolds. Prove that $X \times Y$ is orientable. Conclude that the n-torus $T^n = S^1 \times \cdots \times S^1$ is orientable.

(3) Let A be an $n \times n$ matrix and b a vector in $\mathbb{R}^n$. Show that the map $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(x) = Ax + b$ is a smooth map with $df_x = A$ at every $x \in \mathbb{R}^n$. Conclude that f is a diffeomorphism if and only if $\det(A) \neq 0$.

(4) Let $a \in \mathbb{R}$ be a constant, and define $g_a : \mathbb{R} \to S^1 \times S^1$ by

$$g_a(t) = (\cos t, \sin t, \cos(at), \sin(at)).$$

Prove that g_a is an immersion. For which a is g_a injective and why? For which a is g_a an embedding and why?

(5) Let $f : X \to Y$ be a smooth function. Define $gr^f : X \to X \times Y$ by $gr^f(x) = (x, f(x))$. Prove that gr^f is an embedding. Conclude that its image $graph(f) = gr^f(X)$ is a smooth submanifold of Y which is diffeomorphic to X.

(6) (a) If f and g are immersions, show that $f \times g$ is an immersion.

  (b) If f and g are submersions, show that $f \times g$ is a submersion.

  (c) If f and g are immersions, show that $g \circ f$ is an immersion.

  (d) If f and g are submersions, show that $g \circ f$ is a submersion.