MAT 239: HOMEWORK 4

(1) Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x_1, \ldots, x_n) = \varepsilon_1 x_1^2 + \cdots + \varepsilon_n x_n^2$ where $\varepsilon_i \in \{-1, 1\}$.

  (a) For $n = 2$, sketch $f^{-1}(-1)$, $f^{-1}(0)$, and $f^{-1}(1)$ for the different choices of ε_i.

  (b) Show that 0 is the only critical value of f, and conclude that $f^{-1}(1)$ and $f^{-1}(-1)$ are submanifolds.

  (c) Describe $T_{(x_1,\ldots,x_n)}(f^{-1}(1))$.

(2) Let $SL(n; \mathbb{R})$ denote the set of $n \times n$ matrices of determinant 1.

  (a) Prove that $SL(n; \mathbb{R})$ is a manifold of dimension $n^2 - 1$. [Hint: Show that if $\det(A) \neq 0$, det is a submersion even when restricted to the set $\{tA \mid t > 0\} \subset M_{n \times n}$.]

  (b) Let I be the identity matrix. Prove that $T_I(SL(n; \mathbb{R}))$ is all matrices of trace zero.

(3) Let $p : \mathbb{C} \to \mathbb{C}$ be a complex polynomial of one variable $z = x + iy$, $p(z) = z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$. Prove that p has finitely many critical points.

(4) Let $L : \mathbb{C}^2 \to \mathbb{C}$ be the complex map $L(z_1, z_2) = z_1 z_2$ where $z_j = x_j + i y_j$ and we use complex multiplication. Breaking up the function into real components, show that $0 \in \mathbb{C}$ is the only critical value, so in particular, $L^{-1}(1)$ is a submanifold. What manifold is $L^{-1}(1)$ diffeomorphic to? What does $L^{-1}(0)$ look like?

(5) Consider the function $f : S^2 \to \mathbb{R}$ defined by $f(x, y, z) = z$ for $(x, y, z) \in S^2$. Find all the critical points, critical values, and regular values of f.

(6) Is the intersection of these two submanifolds transverse in $\mathbb{R}^3$? Explain why or why not.

  (a) $\{z = x^2 + y^2\}$ with $\{z = 0\}$ in $\mathbb{R}^3$.

  (b) $\{x = 0\}$ with $\{z = 0\}$ in $\mathbb{R}^3$.

  (c) $\{x = y = 0\}$ with $\{x = z = 0\}$ in $\mathbb{R}^3$.

  (d) $\{x = y = 0\}$ with $\{x = z = 1\}$ in $\mathbb{R}^3$.

  (e) $\{x = y = 0\}$ with $\{z = x + y\}$ in $\mathbb{R}^3$.

  (f) $\{x = y = 0\}$ with $\{x = y^2 + z^2\}$ in $\mathbb{R}^3$.

(7) Let $f : X \to Y$ be a smooth map transverse to a submanifold $Z \subset Y$. Let $W = f^{-1}(Z)$. (We know that W is a submanifold of X.) Prove that $T_x W$ is the preimage of $T_{f(x)}(Z) \subset T_{f(x)} Y$ under the linear map $df_x : T_x X \to T_{f(x)} Y$.