**MAT 239: HOMEWORK 5**

(1) Let $f : \mathbb{R}^k \to \mathbb{R}$ be a smooth function. For each $a \in \mathbb{R}^k$ define
$$f_a(x_1, \ldots, x_k) = f(x) + a_1 x_1 + \cdots + a_k x_k.$$
Prove that for almost every $a \in \mathbb{R}^k$ (in the sense of Sard's theorem), that $f_a$ is a Morse function.

Hint: show that you can apply the generic transversality theorem to the family of functions $F : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ defined by
$$F(x, a) = \left( \frac{\partial f}{\partial x_1}(x) + a_1, \ldots, \frac{\partial f}{\partial x_k} + a_k \right)$$

(2) Let $E = \{x^2 + 2y^2 + 3z^2 = 1\}$ and let $S_a = \{x^2 + y^2 + z^2 = a\}$. Determine for which values of $a$, $E$ intersects $S$ transversally.

(3) Each of the following functions has a critical point a the origin. Determine whether or not this critical point is non-degenerate, and if it is, determine whether it is a local maximum (index 2), saddle (index 1), or local minimum (index 0).

(a) $f(x, y) = x^2 + 4y^3$

(b) $f(x, y) = x^2 - 2xy + y^2$

(c) $f(x, y) = x^2 + y^4$

(d) $f(x, y) = x^2 + 11xy + y^2/2 + x^6$

(e) $f(x, y) = 10xy + y^2 + 75y^3$