MAT 239: HOMEWORK 6

(1) Let $f : X \to Y$ and $g : Y \to Z$ be smooth maps between manifolds, and suppose X is compact. Further suppose that $\dim(X) + \dim(W) = \dim(Z)$. Assume g is transverse to a submanifold W of Z so $g^{-1}(W)$ is a submanifold of Y. Prove that
$$I_2(f, g^{-1}(W)) = I_2(g \circ f, W).$$

(2) Let $\Delta \subset S^1 \times S^1$ be the diagonal submanifold $\Delta = \{(e^{i\theta}, e^{i\theta}) \subset S^1 \times S^1\}$. We will calculate $I_2(\Delta, \Delta)$.

   (a) Show that $f : S^1 \to S^1 \times S^1$ defined by $f(e^{i\theta}) = (e^{i\theta}, e^{i(\theta+\varepsilon)})$ is homotopic to the inclusion of the diagonal $i : S^1 \to S^1 \times S^1$ $i(e^{i\theta}) = (e^{i\theta}, e^{i\theta})$.

   (b) Show that for $0 < \varepsilon < 2\pi$, the above defined map f is transverse to Δ.

   (c) Calculate $I_2(\Delta, \Delta)$ by counting the number of points in $f^{-1}(\Delta)$ mod 2.

(3) If $f : X \to Y$ is homotopic to a constant map and $\dim X > 0$, show that for any submanifold $Z \subset Y$ such that $\dim X + \dim Z = \dim Y$, $I_2(f, Z) = 0$.

(4) Let $f_0, f_1 : X \to \mathbb{R}^n$ be two smooth maps. The *straight line homotopy* from f_0 to f_1 is defined by $F : X \times I \to \mathbb{R}^n$
$$F(x, t) = (1 - t)f_0(x) + tf_1(x).$$
(Note, it is important that the target space is $\mathbb{R}^n$ so we have the vector space operations of scalar multiplication and addition.)

   (a) Check that F is indeed a smooth homotopy from f_0 to f_1.

   (b) Prove that every map $f : X \to \mathbb{R}^n$ is homotopic to a constant map.

   (c) Conclude using the previous problem that for any map $f : X \to \mathbb{R}^n$ with $\dim X > 0$ and any submanifold $Z \subset \mathbb{R}^n$ with $\dim X + \dim Z = n$ that $I_2(f, Z) = 0$.

(5) Two compact submanifolds $K, L \subset M$ are cobordant in M if there exists a compact submanifold with boundary W in $M \times I$ (where $\partial W = W \cap \partial(M \times I)$) such that $W \cap M \times \{0\} = K \times \{0\}$ and $W \cap M \times \{1\} = L \times \{1\}$. (Note that this means $\dim K = \dim L = \dim W - 1$.) Prove that if K and L are cobordant in M then for every submanifold Z of M such that $\dim K + \dim Z = \dim M$, $I_2(K, Z) = I_2(L, Z)$.

(6) So far we have defined the (mod 2) intersection number between two submanifolds, or between a map and a submanifold of its target space. In this problem we will define the (mod 2) intersection number between two maps. Suppose X and Z are compact manifolds, and $f : X \to Y$ and $g : Z \to Y$ are smooth maps into a manifold Y. Suppose further that $\dim X + \dim Z = \dim Y$. Let $\Delta = \{(y, y) \in Y \times Y\}$ be the diagonal submanifold. Define
$$I_2(f, g) := I_2(f \times g, \Delta).$$

   (a) Check that the dimensions add up appropriately for $I_2(f \times g, \Delta)$ to make sense.

   (b) Prove that $I_2(f, g) = I_2(f', g) = I_2(f, g')$ if f is homotopic to f' and g is homotopic to g'.

   (c) Prove that $I_2(f, g) = I_2(g, f)$. (Hint: Use the map $s : Y \times Y \to Y \times Y$ defined by $s(a, b) = (b, a)$, and the first exercise.)

   (d) In the case that Z is a submanifold of Y and g is the inclusion map $g = i : Z \to Y$, check that $I_2(f, i) = I_2(f, Z)$. (Check that the new definition gives the same answer as the old definition.)

   (e) Use the previous part to prove that if X, Z are compact submanifolds of Y then $I_2(X, Z) = I_2(Z, X)$.