MAT 239: HOMEWORK 6

(1) Let $C_{p,q}$ be the curve in the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by the points

$$C_{p,q} = \{(qt, pt) \mid t \in \mathbb{R}\}.$$

Orient $C_{p,q}$ by the image of the positive ∂_t vector under the differential of the immersion $\phi(t) = (qt, pt)$.

(a) Calculate the oriented intersection number $I(C_{1,0}, C_{p,q})$.

(b) Calculate the oriented intersection number $I(C_{0,1}, C_{p,q})$.

(c) Calculate the oriented intersection number $I(C_{3,5}, C_{p,q})$.

(d) (Ungraded) Can you find a general formula for $I(C_{m,n}, C_{p,q})$?

(2) Recall that the *degree* of a map $f : X \to Y$ where $\dim X = \dim Y$ is the oriented intersection number of f with a point $\{y\}$. For a positive integer m prove that

(a) $f_m : S^1 \to S^1$ defined by $f_m(z) = z^m$ (viewing $z \in S^1 \subset \mathbb{C}$ as a complex number) has degree m.

(b) $g_m : S^1 \to S^1$ defined by $g_m(z) = \overline{z}^m$ has degree $-m$ (where the bar denotes complex conjugation).

(3) The antipodal map $a : S^k \to S^k$ is defined by $a(x) = -x$.

(a) Compute the degree of a in terms of k.

(b) Prove that the antipodal map is homotopic to the identity if and only if k is odd. Hint: To construct the homotopy when k is odd consider the following homotopy in the $k = 1$ case:

$$\begin{bmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{bmatrix}$$

(4) Euler characteristic and vector fields. Let $\Delta \subset X \times X$ be the diagonal submanifold of points $\{(x, x)\}$.

(a) Show that the tangent bundle is

$$T_{(x,x)}(\Delta) = \{(v, v) \in T_xX \times T_xX \cong T_{(x,x)}(X \times X)\}$$

and the normal bundle is

$$N_{(x,x)}(\Delta; X \times X) = \{(v, -v) \in T_xX \times T_xX \cong T_{(x,x)}(X \times X)\}.$$

(b) Prove that the map $\Psi : TX \to N(\Delta; X \times X)$ defined by $\Psi(x, v) = ((x, x), (v, -v))$ is a diffeomorphism. By the tubular neighborhood theorem, this implies there is a neighborhood of X in TX, a neighborhood of Δ in $X \times X$, and a diffeomorphism between these neighborhoods which sends X to Δ in the usual way $x \mapsto (x, x)$.

(c) Let Z denote the zero section $Z = \{(x, 0) \in TX\}$. Prove that the self-intersection number of the diagonal $I(\Delta, \Delta)$ in $X \times X$ (i.e. the Euler characteristic of X) is equal to the self-intersection number of the zero section $I(Z, Z)$ in TX. (Orient TX by locally in each coordinate chart specifying a basis for $T_{(x,v)}(TX)$ of the form $(\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n})$ where $(x_1, \ldots, x_n)$ give coordinates on X and $(y_1, \ldots, y_n)$ are the corresponding induced coordinates on T_xX.

(d) Let $V : X \to TX$ be any vector field on X. Prove that $I(V, Z) = I(Z, Z)$ by constructing a homotopy from V to the inclusion of the zero section. Conclude that the Euler characteristic of X is equal to $I(V, Z)$ for any vector field V on X.

(5) Morse theory and Euler characteristic.

(a) Let $f_k(x_1, \ldots, x_n) = -x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots x_n^2$ be the standard model for an index k Morse critical point. Calculate the gradient vector field

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \right\rangle.$$

(b) Let Z denote the zero section $Z = \{((x_1, \ldots, x_n), (0, \ldots, 0)) \in T\mathbb{R}^n\}$. Prove that ∇f is transverse to Z.

(c) Calculate $I(\nabla f, Z)$ in terms of k.

(d) Suppose $f : X \to \mathbb{R}$ is a Morse function. By the Morse Lemma, for every critical point of f, there exist local coordinates defined near that point such that f looks like one of the standard models in part (a). The corresponding value of k is called the *index* of the critical point of f. Let $k(p)$ denote the index of a critical point $p \in Crit(f)$. Prove that

$$I(\nabla f, Z) = \sum_{p \in Crit(f)} (-1)^{k(p)}.$$

(By the previous problem, this computes the Euler characteristic of X. Note that a key idea in Morse theory is that a Morse function on X induces a cell decomposition where critical points of index k are in bijection with cells of dimension k. This recovers the definition of Euler characteristic as an alternating sum of the number of cells.)

(6) Let X be a manifold with boundary, Y a manifold, and Z a submanifold. Fix orientations on X, Y, and Z. Let $f : X \to Y$ be a smooth map such that f and ∂f are transverse to Z. Then $W := f^{-1}(Z)$ is a submanifold with boundary of X. Let A denote the boundary of $f^{-1}(Z)$. Show that the following two induced orientations on A differ by $(-1)^{\operatorname{codim} Z}$:

- First give $W = f^{-1}(Z)$ the preimage orientation induced by f. Then give A the boundary orientation induced as the boundary of W.
- First give ∂X the boundary orientation induced as the boundary of X. Then give A the preimage orientation induced as the preimage of $\partial f : \partial X \to Y$.

Hint: At a fixed point $p \in A$, show that

(a) you can choose the same vector n_p to serve as an outward pointing vector in $T_p(W)$ to A, and as an outward pointing vector in $T_p(X)$ to ∂X.

(b) you can choose the same basis $(v_1, \ldots, v_k)$ for the normal space $N_p(A; \partial X)$ to serve also as a basis for $N_p(W; X)$.

Then name a basis $(w_1, \ldots, w_\ell)$ for T_pA and an oriented basis $(u_1, \ldots, u_m)$ for $T_{f(p)}Z$, and use the definitions of induced orientations to determine what condition ensures that $(w_1, \ldots, w_\ell)$ represents the above specified orientation in each of the two cases.