MAT 239: HOMEWORK 8

(1) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism. Put coordinates (x_1, x_2, x_3) on the domain and (y_1, y_2, y_3) on the co-domain. Show directly from the definition of the pull-back (compute explicitly) that

$$f^*(dy_1 \wedge dy_2 \wedge dy_3)_p = \det(df_p) dx_1 \wedge dx_2 \wedge dx_3.$$

(2) Let $f : X \to Y$ be a smooth map between manifolds, and let $\phi : Y \to \mathbb{R}$ be a smooth function (i.e. a zero form on Y). Then the derivative $d\phi$ can be viewed as a 1-form on Y. Prove directly using the definiions that

$$f^*(d\phi) = d(f^*\phi).$$

(3) Let $c : [a, b] \to X$ be a smooth curve, and let $c(a) = p$ and $c(b) = q$. Show that if $\omega = df$ where $f : X \to \mathbb{R}$ is a smooth function then

$$\int_a^b c^*\omega = f(q) - f(p)$$

(4) Let $\omega \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$ be defined by

$$\omega_{(x,y)} = \left(\frac{-y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy.$$

Let $C_R \subset \mathbb{R}^2 \setminus \{0\}$ the the circle of radius R. Calculate $\int_{C_R} \omega$.

(5) Let $\alpha = dz - \sum_{i=1}^n y_i dx_i$ on $\mathbb{R}^{2n+1}$. Prove that the $2n + 1$ form $\omega := \alpha \wedge (d\alpha)^n$ is non-zero at every point in $\mathbb{R}^{2n+1}$.

(6) Prove the following properties for the exterior derivative d defined on a smooth manifold (you may use the corresponding properties for the exterior derivative on $\mathbb{R}^n$):

(a) $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge (d\eta)$ where $\omega \in \Omega^p(X)$.

(b) $d(d\omega) = 0$.

(c) $d \circ g^* = g^* \circ d$.

(7) (Not graded, but highly recommended) Work through the exercises in Guillemin Pollack Chapter 4 Section 6 (Cohomology wih Forms).