SHORT CALCULUS Math 16C Sec 2 Spring 2008
Homework #1 Solutions
Peter Malkin

Section C.1

Question 28

Since $y = x(\ln x + C)$, we have $\frac{dy}{dx} = (\ln x + C) + x(\frac{1}{x}) = \ln x + C + 1$. Thus,

$$\begin{aligned} x + y - xy' &= x + [x(\ln x + C)] - x[\ln x + C + 1] \\ &= x + x\ln x + Cx - x\ln x - Cx - x \\ &= 0 \end{aligned}$$

as required.

Question 46

Since $y = Ce^{x-x^2}$, we have $y' = C(1-2x)e^{x-x^2}$.

$$\begin{aligned} y' + (2x-1)y &= [C(1-2x)e^{x-x^2}] + (2x-1)[Ce^{x-x^2}] \\ &= C(1-2x)e^{x-x^2} - C(1-2x)e^{x-x^2} \\ &= 0 \end{aligned}$$

as required.

Then, since $y = 2$ when $x = 1$, we have $2 = Ce^{1-1^2} = C \Leftrightarrow C = 2$.

So, the particular solution is $\boxed{y = 2e^{x-x^2}}$.

Section C.2

Question 10

$$\begin{aligned}
&\frac{dy}{dx} = x^2 y \\
\Leftrightarrow\ &\frac{1}{y}dy = x^2 dx \\
\Leftrightarrow\ &\int \frac{1}{y}dy = \int x^2 dx \\
\Leftrightarrow\ &\ln|y| = \frac{1}{3}x^3 + C \text{ (C is a constant)} \\
\Leftrightarrow\ &|y| = Ae^{\frac{x^3}{3}} \text{ (A is a constant)} \\
\Leftrightarrow\ &y = Ae^{\frac{x^3}{3}}
\end{aligned}$$

The general solution is $\boxed{y = Ae^{\frac{x^3}{3}}}$.

Question 22

$$\begin{aligned}
&\frac{dy}{dx} - y(x+1) = 0 \\
\Leftrightarrow\ &\frac{dy}{dx} = y(x+1) \\
\Leftrightarrow\ &\frac{1}{y}dy = (x+1)dx \\
\Leftrightarrow\ &\int \frac{1}{y}dy = \int (x+1)dx \\
\Leftrightarrow\ &\ln|y| = \frac{1}{2}x^2 + x + C \text{ (C is a constant)} \\
\Leftrightarrow\ &|y| = e^{(\frac{1}{2}x^2+x+C)} \\
\Leftrightarrow\ &y = Ae^{(\frac{1}{2}x^2+x)} \text{ (A is a constant)}
\end{aligned}$$

The general solution is $\boxed{y = Ae^{(\frac{1}{2}x^2+x)}}$.

Question 30

$$\begin{aligned}
\frac{dy}{dx} &= x^2(1+y) \\
\frac{1}{1+y}dy &= x^2 dx \\
\int \frac{1}{1+y}dy &= \int x^2 dx \\
\ln|1+y| &= \frac{1}{3}x^3 + C \text{ (C is a constant)} \\
|1+y| &= e^{\frac{x^3}{3}+C} \\
|1+y| &= Ae^{\frac{x^3}{3}} \text{ (A is a constant)} \\
1+y &= Ae^{\frac{x^3}{3}} \\
y &= Ae^{\frac{x^3}{3}} - 1
\end{aligned}$$

The general solution is $\boxed{y = Ae^{\frac{x^3}{3}} - 1}$.

Since $y = 3$ when $x = 0$, we have $3 = Ae^0 - 1 \Leftrightarrow A = 4$. Thus, the particular solution is $\boxed{y = 4e^{\frac{x^3}{3}} - 1}$.

Question 42

First, we compute the general solution of the differential equation.

$$\begin{aligned}
\frac{dS}{dt} &= \frac{kS}{t^2} \\
\Leftrightarrow \frac{1}{S}dS &= \frac{k}{t^2}dt \\
\Leftrightarrow \int \frac{1}{S}dS &= \int \frac{k}{t^2}dt \\
\Leftrightarrow \ln(S) &= -\frac{k}{t} + C \text{ (C is a constant) ($S \geq 0$)} \\
\Leftrightarrow S &= e^{-\frac{k}{t}+C} \\
\Leftrightarrow S &= Ae^{-\frac{k}{t}} \text{ (A is a constant)}
\end{aligned}$$

The general solution is $\boxed{S = Ae^{-\frac{k}{t}}}$.

Second, we find the particular solution using initial conditions. We are given that $50 = \lim_{t\to\infty} S$. Now,

$$\lim_{t\to\infty} S = \lim_{t\to\infty} Ae^{-\frac{k}{t}} = Ae^0 = A.$$

So, $A = 50$. Thus, the equation now is $S = 50e^{-\frac{k}{t}}$. We are also given that $S = 10$ when $t = 1$. So,

$$10 = 50e^{-\frac{k}{1}} \Leftrightarrow \frac{1}{5} = e^{-k} \Leftrightarrow \ln(\frac{1}{5}) = -k \Leftrightarrow k = -\ln(\frac{1}{5}) = \ln(5).$$

Thus, the particular solution is $\boxed{S = 50e^{-\frac{\ln(5)}{t}}}$.

Section C.3

Question 10

Step 1: Put equation into standard form.

$$\frac{dy}{dx} + \underbrace{3}_{P(x)} y = \underbrace{e^{-3x}}_{Q(x)}.$$

Step 2: Compute the integrating factor.

$$u(x) = e^{\int P(x)dx} = e^{\int 3dx} = e^{3x}.$$

Step 3: Compute the general solution.

$$y = \frac{1}{u(x)} \int Q(x)u(x)dx = \frac{1}{e^{3x}} \int e^{-3x}e^{3x}dx = e^{-3x} \int 1dx = e^{-3x}\left(x + C\right).$$

where C is a constant.

Thus, the general solution is $\boxed{y = e^{-3x}\left(x + C\right)}$.

Question 12

Step 1: Put equation into standard form.

$$\frac{dy}{dx} + \underbrace{\frac{2}{x}}_{P(x)} y = \underbrace{(3x+1)}_{Q(x)}.$$

Step 2: Compute the integrating factor.

$$u(x) = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln(x^2)} = x^2.$$

Step 3: Compute the general solution.

$$y = \frac{1}{u(x)} \int Q(x)u(x)dx = \frac{1}{x^2} \int (3x+1)x^2dx = x^{-2} \int 3x^3 + x^2dx = x^{-2}\left(\frac{3}{4}x^4 + \frac{1}{3}x^3 + C\right)$$

where C is a constant.

Thus, the general solution is $\boxed{y = \frac{3}{4}x^2 + \frac{1}{3}x + Cx^{-2}}$.

Question 32

Step 1: Put the equation into standard form.

$$\frac{dy}{dx} + \underbrace{(2x-1)}_{P(x)} y = \underbrace{0}_{Q(x)}.$$

Step 2: Compute the integrating factor.

$$u(x) = e^{\int P(x)dx} = e^{\int (2x-1)dx} = e^{x^2-x}.$$

Step 3: Compute the general solution.

$$y = \frac{1}{u(x)} \int Q(x)u(x)dx = \frac{1}{e^{x^2-x}} \int 0(e^{x^2-x})dx = e^{x-x^2} \int 0dx = e^{x-x^2}C$$

where C is a constant.

Thus, the general solution is $\boxed{y = Ce^{x-x^2}}$.

Since $y = 2$ when $x = 1$, we have $2 = Ce^{1-1^2} \Leftrightarrow 2 = C$. Thus, the particular solution is $\boxed{y = 2e^{x-x^2}}$.

Question 44

First, note that $0 \le y \le 1$.

Step 1: Put the equation into standard form.

$$\frac{dy}{dx} = \frac{1-y}{4} \Leftrightarrow \frac{dy}{dx} = \frac{1}{4} - \frac{1}{4}y \Leftrightarrow \frac{dy}{dx} + \underbrace{\frac{1}{4}}_{P(t)} y = \underbrace{\frac{1}{4}}_{Q(t)}.$$

Step 2: Compute the integrating factor.

$$u(t) = e^{\int P(t)dt} = e^{\int \frac{1}{4}dt} = e^{\frac{t}{4}}.$$

Step 3: Compute the general solution.

$$y = \frac{1}{u(x)} \int Q(x)u(x)dx = \frac{1}{e^{\frac{t}{4}}} \int \frac{1}{4} e^{\frac{t}{4}} dx = e^{-\frac{t}{4}} \left(e^{\frac{t}{4}} + C \right) = 1 + Ce^{-\frac{t}{4}}$$

where C is a constant.

Thus, the general solution is $\boxed{y = 1 + Ce^{-\frac{t}{4}}}$.

Part (a): Since $y = 0$ when $t = 0$, we have $0 = 1 + C^0 \Leftrightarrow C = -1$. Thus, the particular solution is $\boxed{y = 1 - e^{-\frac{t}{4}}}$.

Part (b): We must find t when $y = 0.5$.

$$0.5 = 1 - e^{-\frac{t}{4}} \Leftrightarrow 0.5 = e^{-\frac{t}{4}} \Leftrightarrow \ln(0.5) = -\frac{t}{4} \Leftrightarrow t = -4\ln(0.5) = 4\ln(2).$$

Thus, half the polutation has been exposed after $t = 4\ln(2) \approx 2.77$ years.

Part (c): We must find y when $t = 4$.

$$y = 1 - e^{-\frac{4}{4}} = 1 - e^{-1} \approx 0.632.$$

So, after 4 years approximately 63% of the population has been exposed.