SHORT CALCULUS Math 16C Sec 2 Spring 2008
Homework #4 Solutions
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Section 7.6

Question 2

Let $F(x, y, \lambda) = xy - \lambda(2x + y - 4)$.

$$F_x = y - 2\lambda = 0 \Leftrightarrow \lambda = \frac{1}{2}y \tag{1}$$
$$F_y = x - \lambda = 0 \Leftrightarrow \lambda = x \tag{2}$$
$$F_\lambda = -(2x + y - 4) = 0 \tag{3}$$

Using equation (1) and (2) gives $\frac{1}{2}y = x \Rightarrow y = 2x$. Putting this into equation (3) gives

$$2x + 2x - 4 = 0 \Rightarrow x = 4,$$

so $y = 2$. Thus, $f(x, y) = xy$ is maximum at the point $(1, 2)$, and the maximum value is $f(1, 2) = 2$.

Question 6

Let $F(x, y, \lambda) = x^2 - y^2 - \lambda(x - 2y + 6)$.

$$F_x = 2x - \lambda = 0 \tag{4}$$
$$F_y = -2y + 2\lambda = 0 \tag{5}$$
$$F_\lambda = -(x - 2y + 6) = 0 \tag{6}$$

Equation (1) implies $\lambda = 2x$, and substituting this into (2) gives, $-2y + 4x = 0$, which implies $y = 2x$. Substituting this into (3) gives

$$-(x - 4x + 6) = 0 \Rightarrow x = 2.$$

So, $y = 2x = 4$. Thus, $f(x, y)$ is maximum at $(2, 4)$, and the maximum value is $f(2, 4) = -12$.

Question 16

Let $F(x, y, \lambda) = x^2 - 8x + y^2 - 12y + 48 - \lambda(x + y - 8)$.

$$F_x = 2x - 8 - \lambda = 0 \tag{1}$$
$$F_y = 2y - 12 - \lambda = 0 \tag{2}$$
$$F_\lambda = -(x + y - 8) = 0 \tag{3}$$

$$(1) - (2): \quad 2x - 2y + 4 = 0 \Rightarrow x - y + 2 = 0 \tag{4}$$
$$(4) + (3): \quad -2y + 10 = 0 \Rightarrow y = 5 \tag{5}$$

Substituting $y = 5$ into (3) gives $-(x + 5 - 8) = 0 \Rightarrow x = 3$. So, $f(x, y)$ is maximum at $(3, 5)$, and the maximum value is $f(3, 5) = 3^2 - 8 \cdot 3 + 5^2 - 12 \cdot 5 + 48 = -2$.

Question 18

Let $F(x, y, z, \lambda) = x^2y^2z^2 - \lambda(x^2 - y^2 - z^2 - 1)$.

$$F_x = 2xy^2z^2 - \lambda 2x = 0 \Rightarrow 2x(\lambda - y^2z^2) = 0 \Rightarrow x = 0 \text{ or } \lambda = y^2z^2 \quad (1)$$
$$F_y = 2yx^2z^2 - \lambda 2y = 0 \Rightarrow 2y(\lambda - x^2z^2) = 0 \Rightarrow y = 0 \text{ or } \lambda = x^2z^2 \quad (2)$$
$$F_z = 2zx^2y^2 - \lambda 2z = 0 \Rightarrow 2z(\lambda - x^2y^2) = 0 \Rightarrow z = 0 \text{ or } \lambda = x^2y^2 \quad (3)$$
$$F_\lambda = -(x^2 - y^2 - z^2 - 1) = 0 \quad (4)$$

Note that we can ignore the cases where $x = 0$, $y = 0$, or $z = 0$ since they do not give a maximum. Then, (1) and (2) imply $y^2z^2 = \lambda = x^2z^2 \Rightarrow z = 0$ or $y^2 = x^2 \Rightarrow x = y$ (since $x > 0$ $y > 0$). Then, (1) and (3) imply $y^2z^2 = \lambda = x^2y^2 \Rightarrow y = 0$ or $z^2 = x^2 \Rightarrow z = x$ (since $x > 0$ $z > 0$). Again, we can ignore when $z = 0$ or $y = 0$. Then, substituting $y = x$ and $z = x$ into (4) gives

$$-(x^2 - x^2 - x^2 - 1) = 0 \Rightarrow 3x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{3}}.$$

Note that $x > 0$. So, $x = y = z = \frac{1}{\sqrt{3}}$. So, the maximum of f occurs at $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. The maximum value is $f(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}$.

Question 24

Let $F(x, y, z, \lambda, \mu) = xy + yz - \lambda(x + 2y - 6) - \mu(x - 3z)$.

$$F_x = y - \lambda - \mu = 0 \quad (1)$$
$$F_y = x + z - 2\lambda = 0 \quad (2)$$
$$F_z = y + 3\mu = 0 \quad (3)$$
$$F_\lambda = -(x + 2y + -6) = 0 \quad (4)$$
$$F_\mu = -(x - 3z) = 0 \quad (5)$$

$$(2) - 2 \times (1): \quad x - 2y + z + 2\mu = 0 \quad (6)$$
$$(6) - \frac{2}{3} \times (3): \quad x - \frac{8}{3}y + z = 0 \quad (7)$$
$$(7) + (5): \quad -\frac{8}{3}y + 4z = 0 \Rightarrow -\frac{2}{3}y + z = 0 \quad (8)$$
$$(5) - (4): \quad 2y + 3z - 6 = 0 \quad (9)$$
$$(9) - 3 \times (8): \quad 4y - 6 = 0 \Rightarrow y = \frac{3}{2} \quad (10)$$

Substituting $y = \frac{3}{2}$ into (9) gives $3 + 3z - 6 = 0 \Rightarrow z = 1$. Then, substituting $z = 1$ into (5) gives, $-(x - 3) = 0 \Rightarrow x = 3$.

So, $f(x, y, z)$ is maximum at $(3, \frac{3}{2}, 1)$. The maximum value is $f(3, \frac{3}{2}, 1) = 3 \cdot \frac{3}{2} + \frac{3}{2} \cdot 1 = 6$.

Question 38

We must produce 1000 units. So, $x_1 + x_2 = 1000$. We wish to minimize the cost

$$C(x_1, x_2) = 0.25x_1^2 + 25x_1 + 0.05x_2^2 + 12x_2.$$

Let $F(x_1, x_2, \lambda) = 0.25x_1^2 + 25x_1 + 0.05x_2^2 + 12x_2 - \lambda(x_1 + x_2 - 1000)$.

$$F_{x_1} = 0.5x_1 + 25 - \lambda = 0 \Rightarrow \lambda = 0.5x_1 + 25 \quad (1)$$
$$F_{x_2} = x_2 + 12 - \lambda = 0 \Rightarrow \lambda = 0.1x_2 + 12 \quad (2)$$
$$F_\lambda = -(x_1 + x_2 - 1000) = 0 \quad (3)$$

Using equation (1) and (2) gives

$$0.5x_1 + 25 = 0.1x_2 + 12 \Leftrightarrow x_2 = 5x_1 + 130.$$

Putting this into equation (3) gives

$$x_1 + (5x_1 + 130) - 1000 = 0 \Leftrightarrow 6x_1 = 870 \Leftrightarrow x_1 = 145.$$

So, $x_2 = 5(145) + 130 = 855$.

Thus, 145 units should be produced at location 1 and 855 units at location 2 giving a minimal cost of $C(145, 855) = \$55692.50$

Question 48

The are of a rectangle with dimensions x and y is $f(x, y) = xy$. The perimeter of the rectangle is $x + y = P$. So, we must maximize xy subject to the constraint that $2x + 2y = P$. Let $F(x, y, \lambda) = xy - \lambda(2x + 2y - P)$.

$$F_x = y - \lambda = 0 \Leftrightarrow y = \lambda \quad (1)$$
$$F_y = x - \lambda = 0 \Leftrightarrow x = \lambda \quad (2)$$
$$F_\lambda = -(x + y - P) = 0 \quad (3)$$

Equation (1) and (2) imply that $y = x$. Putting this into equation (3) gives $2x + 2x - P = 0 \Rightarrow x = \frac{1}{4}P$, and so $y = \frac{1}{4}P$. Thus, the maximum area of the rectangle is $(\frac{1}{4}P)^2 = \frac{1}{16}P^2$ as required. Note that the rectangle with maximal area is a square.

Section 7.8

Question 2

$$\int_{x}^{x^2} \frac{y}{x} dy = \left[\frac{1}{2} \cdot \frac{y^2}{x}\right]_{x}^{x^2} = \frac{1}{2}\left(\frac{(x^2)^2}{x} - \frac{(x)^2}{x}\right) = \frac{1}{2}\left(x^3 - x\right).$$

Question 6

$$\begin{aligned}\int_{x^2}^{\sqrt{x}} \left(x^2 + y^2\right) dy &= \left[yx^2 + \frac{1}{3}y^3\right]_{x^2}^{\sqrt{x}} \\ &= \left((\sqrt{x})x^2 + \frac{1}{3}(\sqrt{x})^3\right) - \left((x^2)x^2 + \frac{1}{3}(x^2)^3\right) \\ &= x^{\frac{3}{2}} + \frac{1}{3}x^{\frac{7}{2}} - x^4 - \frac{1}{3}x^6.\end{aligned}$$

Question 12

$$\begin{aligned}\int_0^2 \int_0^2 6 - x^2 dy dx &= \int_0^2 \left[6y - x^2y\right]_0^2 dx \\ &= \int_0^2 (12 - 2x^2) dx \\ &= \left[12x - \frac{2}{3}x^3\right]_0^2 \\ &= 12(2) - \frac{2}{3}(2)^3 = 24 - \frac{16}{3} = \frac{56}{3}\end{aligned}$$

Question 16

$$\begin{aligned}\int_0^2 \int_{3y^2-6y}^{2y-y^2} 3y dx dy &= \int_0^2 \left(\left[3xy\right]_{3y^2-6y}^{2y-y^2}\right) dy \\ &= \int_0^2 \left(3(2y - y^2)y - 3(3y^2 - 6y)y\right) dy = \int_0^2 (24y^2 - 12y^3) dy \\ &= \left[8y^3 - 3y^4\right]_0^2 \\ &= 8(2)^3 - 3(2)^4 = 16\end{aligned}$$

Question 20

$$\begin{aligned}\int_0^4 \int_0^x \frac{2}{(x+1)(y+1)} dy dx &= \int_0^4 \left[\frac{2}{x+1} \ln|y+1|\right]_0^x \\ &= \int_0^4 \left(\frac{2}{x+1} \ln|x+1| - \frac{2}{x+1} \ln|1|\right) dx \\ &= \int_0^4 \frac{2}{x+1} \ln|x+1| dx\end{aligned}$$

Now, (integration by substitution) let $u = \ln|x+1|$; then, $du = \frac{1}{x+1}dx$, and $u = 0$ when $x = 0$ and $u = \ln(5)$ when $x = 5$. So,

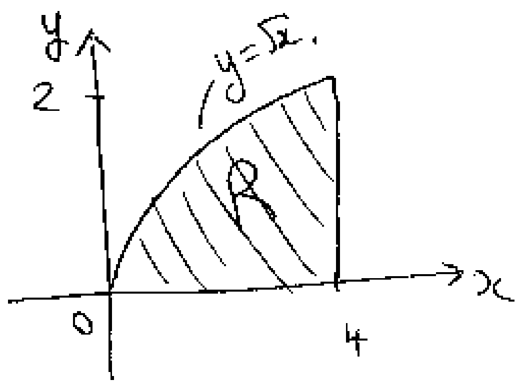
$$\begin{aligned}\int_0^4 \frac{2}{x+1}\ln|x+1|dx &= \int_0^{\ln(5)} 2udu \\ &= \left[u^2\right]_0^{\ln(5)} \\ &= (\ln(5))^2\end{aligned}$$

The above integral can also be solved by the integration by parts technique.

Question 28

The region for which we are computing the area is

$$R = \{(x,y) : 0 \le y \le \sqrt{x}, 0 \le x \le 4\} = \{(x,y) : y^2 \le x \le 4, 0 \le y \le 2\}.$$



So, the are of R is

$$\int_0^4 \int_0^{\sqrt{x}} dydx = \int_0^4 [y]_0^{\sqrt{x}}\, dx = \int_0^4 \sqrt{x}dx = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4 = \frac{2}{3}(4)^{\frac{3}{2}} = \frac{16}{3},$$

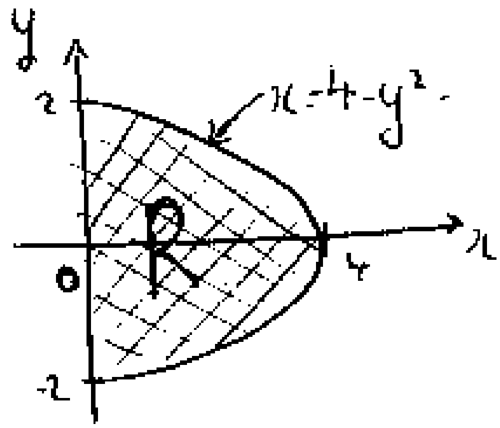
and also, the area of R is

$$\int_0^2 \int_{y^2}^4 dxdy = \int_0^2 [x]_{y^2}^4\, dy = \int_0^2 4 - y^2 dy = \left[4y - \frac{1}{3}y^3\right]_0^2 = 4(2) - \frac{1}{3}(2)^3 = 8 - \frac{8}{3} = \frac{16}{3}.$$

Question 32

The region for which we are computing the area is

$$R = \{(x,y) : -2 \le y \le 2, 0 \le x \le 4 - y^2\} = \{(x,y) : 0 \le x \le 4, -\sqrt{4-x} \le y \le +\sqrt{4-x}\}.$$



So, the area of R is

$$\int_{-2}^{2}\int_{0}^{4-y^2} dxdy = \int_{-2}^{2}[x]_0^{4-y^2}\,dy = \int_{-2}^{2}(4-y^2)dy = \left[4y-\frac{1}{3}y^3\right]_{-2}^{2} = \left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right) = \frac{32}{3},$$

and also, the area of R is

$$\int_{0}^{4}\int_{-\sqrt{4-x}}^{\sqrt{4-x}} dydx = \int_{0}^{4}\left((\sqrt{4-x})-(-\sqrt{4-x})\right)dx = \int_{0}^{4}2\sqrt{4-x}\,dx = \left[-\frac{4}{3}(4-x)^{\frac{3}{2}}\right]_{0}^{4} = \frac{4}{3}4^{\frac{2}{3}} = \frac{32}{3}.$$

Question 38

The region is $R = \{(x,y) : -2 \le x \le 1, x+2 \le y \le 4-x^2\}$. The area of this region is

$$\begin{aligned}
\int_{-2}^{1}\int_{x+2}^{4-x^2} dydx &= \int_{-2}^{1}[y]_{x+2}^{4-x^2}\,dx \\
&= \int_{-2}^{1}\left((4-x^2)-(x+2)\right)dx \\
&= \int_{-2}^{1}(2-x-x^2)dx \\
&= \left[2x-\frac{1}{2}x^2-\frac{1}{3}x^3\right]_{-2}^{1} \\
&= \left(2-\frac{1}{2}-\frac{1}{3}\right)-\left(-4-2+\frac{8}{3}\right) \\
&= \frac{9}{2}
\end{aligned}$$