SHORT CALCULUS Math 16C Sec 2 Spring 2008
Homework #7 Solutions
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Section 10.4

Question 4

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(n-1)!} = -1 + x^2 - \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^5}{24} + ...$$

Question 10

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{3^{n+1}/(n+1)!}{3^n/n!} \cdot \frac{x^{n+1}}{x^n} \right| &= \lim_{n\to\infty} \left| \frac{3^{n+1}}{3^n} \cdot \frac{n!}{(n+1)!} \cdot x \right| \\
&= \lim_{n\to\infty} \left| 3 \cdot \frac{1}{n+1} x \right| \\
&= 0
\end{aligned}$$

By the ratio test, this series converges for all x, so the radius of convergence is ∞.

Question 14

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)!/3^{n+1}}{(-1)^n n!/3^n} \cdot \frac{(x-4)^{n+1}}{(x-4)^n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{(n+1)!}{n!} \cdot \frac{3^n}{3^{n+1}} \cdot (x-4) \right| \\
&= \lim_{n\to\infty} \left| (-1) \cdot (n+1) \cdot \frac{1}{3} \cdot (x-4) \right| \\
&= \infty \qquad (x \neq 4)
\end{aligned}$$

By the ratio test, this series converges for all $x \neq 4$, so the radius of convergence is 0.

Question 16

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{1/\left((n+2)3^{n+2}\right)}{1/\left((n+1)3^{n+1}\right)} \cdot \frac{(x-2)^{n+2}}{(x-2)^{n+1}} \right| &= \lim_{n\to\infty} \left| \frac{n+1}{n+2} \cdot \frac{3^{n+1}}{3^{n+2}} \cdot (x-2) \right| \\
&= \lim_{n\to\infty} \left| \frac{1+\frac{1}{n}}{1+\frac{2}{n}} \cdot \frac{1}{3} \cdot (x-2) \right| \\
&= \left| \frac{1}{3}(x-2) \right|
\end{aligned}$$

By the ratio test, this series converges for all x such that $\left|\frac{1}{3}(x-2)\right| < 1 \Leftrightarrow |x-2| < 3$, so the radius of convergence is 3.

Question 20

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(-1)^{n+2}/(2(n+1)-1)}{(-1)^{n+1}/(2n-1)}\cdot\frac{x^{2(n+1)-1}}{x^{2n-1}}\right| &= \lim_{n\to\infty}\left|\frac{2n-1}{2n+1}\cdot x^2\right| \\ &= \lim_{n\to\infty}\left|\frac{2-\frac{1}{n}}{2+\frac{1}{n}}\cdot x^2\right| \\ &= \left|x^2\right|\end{aligned}$$

By the ratio test, this series converges for all x such that $\left|x^2\right| < 1 \to |x| < 1$, so the radius of convergence is 1.

Question 24

First, note that $\sum_{n=0}^{\infty}\frac{n!x^n}{(n+1)!} = \sum_{n=0}^{\infty}\frac{x^n}{n+1}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{1/(n+1+1)}{1/(n+1)}\cdot\frac{x^{n+1}}{x^n}\right| &= \lim_{n\to\infty}\left|\frac{1/(n+1+1)}{1/(n+1)}\cdot\frac{x^{n+1}}{x^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{n+1}{n+2}\cdot x\right| \\ &= \lim_{n\to\infty}\left|\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\cdot x\right| \\ &= \left|\frac{1+0}{1+0}\cdot x\right| \\ &= |x|\end{aligned}$$

So, by the ratio test, this series converges if $|x| < 1 \Rightarrow -1 < x < 1$. The radius of convergence is thus 1.

Question 26

- $f(x) = e^{-x}$, so $f(0) = 1$.
- $f^{(1)}(x) = -e^{-x}$, so $f^{(1)}(0) = -1$.
- $f^{(2)}(x) = e^{-x}$, so $f^{(2)}(0) = 1$.
- $f^{(3)}(x) = -e^{-x}$, so $f^{(3)}(0) = -1$.
- $f^{(n)}(x) = (-1)^n e^{-x}$, so $f^{(n)}(0) = (-1)^n$.

So, the Taylor Series for e^{-x} is

$$T(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n.$$

Next, we compute the radius of convergence.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(-1)^{n+1}/(n+1)!}{(-1)^n/n!}\cdot\frac{x^{n+1}}{x^n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\cdot\frac{n!}{(n+1)!}\cdot x\right| \\ &= \lim_{n\to\infty}\left|(-1)\cdot\frac{1}{n+1}\cdot x\right| \\ &= 0\end{aligned}$$

So, by the ratio test, this series converges for all x, so its radius of convergence is ∞.

Question 44

The power series for e^x is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

and the power series for e^{-x} is then

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + ...$$

by substituting $-x$ for x in the power series for e^x.

So, the power series for $f(x) = \frac{1}{2}(e^x + e^{-x})$ is then

$$\frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right) = \frac{1}{2}\left(\sum_{n=0}^{\infty}\left(\frac{x^n}{n!} + \frac{(-1)^n x^n}{n!}\right)\right) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{2x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

Question 46

The power series representation of $\frac{1}{x+1}$ is $\frac{1}{x+1} = \sum_{n=0}^{\infty}(-1)^n x^n$. If we multiply this power series by $2x$, then we arrive at the power series for $\frac{2x}{x+1}$. So,

$$\frac{2x}{x+1} = (2x)\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(2x)(-1)^n x^n = \sum_{n=0}^{\infty} 2(-1)^n x^{n+1}.$$

Question 48

The power series representation of $\frac{1}{x+1}$ is $\frac{1}{x+1} = \sum_{n=0}^{\infty}(-1)^n x^n$. If we integrate this power series term by term, then we arrive at the power series for $ln(x+1)$ since $\int \frac{1}{1+x}dx = ln(x+1)$. So,

$$ln(x+1) = \int \frac{1}{1+x}dx = \int \left(\sum_{n=0}^{\infty}(-1)^n x^n\right) dx = \sum_{n=0}^{\infty}\left(\int (-1)^n x^n dx\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}.$$

Question 52

The power series for e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. So,

$$\frac{d}{dx}\left[e^x\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right] = \sum_{n=0}^{\infty}\left(\frac{d}{dx}\left[\frac{x^n}{n!}\right]\right) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Section 10.5

Question 2

The power series for $ln(x)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot (x-1)^{n+1}}{n+1}$, so the power series for $ln(x+1)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{n+1}}{n+1}$, So, the Taylor polynomials of degrees 1, 2, 3, and 4 are as follows:

1. $S_1(x) = x$.

2. $S_2(x) = x - \frac{x^2}{2}$.
3. $S_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$.
4. $S_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$.

Question 6

The Taylor series of $f(x) = \frac{4}{x+1}$ is $\sum_{n=0}^{\infty} 4(-1)^n x^n$. So, the Taylor polynomials of degrees 1, 2, 3, and 4 are as follows:

1. $S_1(x) = 4 - 4x$.
2. $S_2(x) = 4 - 4x + 4x^2$.
3. $S_3(x) = 4 - 4x + 4x^2 - 4x^3$.
4. $S_4(x) = 4 - 4x + 4x^2 - 4x^3 + 4x^4$.

Question 18

The power series for $f(x) = x^2 e^{-x}$ centered at 0 can be obtained by multiplying the power series for e^{-x} by x^2. So,

$$x^2 e^{-x} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+2}.$$

Then,

$$S_6(x) = x^2 - x^3 + \frac{x^4}{2} - \frac{x^5}{6} + \frac{x^6}{24}.$$

Thus,

$$f(\frac{1}{4}) \approx \frac{1}{16} - \frac{1}{64} + \frac{1}{512} - \frac{1}{6144} + \frac{1}{98304} \approx 0.0487.$$

Question 22

The power series for $ln(x)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot (x-1)^{n+1}}{n+1}$, so the power series for $ln(x+1)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{n+1}}{n+1}$, and thus, the power series for $ln(x^2+1)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+2}}{n+1}$.

The 6th degree Taylor polynomial centered at 0 for the function $f(x) = ln(x^2+1)$ is therefore $x^2 - \frac{x^4}{2} + \frac{x^6}{3}$. Thus, we can approximate the following integral:

$$\int_{-\frac{1}{4}}^{\frac{1}{4}} ln(x^2+1) \approx \int_{-\frac{1}{4}}^{\frac{1}{4}} \left(x^2 - \frac{x^4}{2} + \frac{x^6}{3}\right) dx = \left[\frac{x^3}{3} - \frac{x^5}{10} + \frac{x^7}{21}\right]_{-\frac{1}{4}}^{\frac{1}{4}} \approx 0.4873$$