① A) $A = U\Sigma V^T = \boxed{\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\sqrt6&0&0\\0&\sqrt5&0\end{bmatrix}\begin{bmatrix}1/\sqrt6&2/\sqrt6&-1/\sqrt6\\0&1/\sqrt5&2/\sqrt5\\5/\sqrt{30}&-2/\sqrt{30}&1/\sqrt{30}\end{bmatrix}}$

B) $A^+ = V\Sigma^+U^T = \boxed{\begin{bmatrix}1/6&0\\2/6&1/5\\-1/6&2/5\end{bmatrix}}$

C) $\hat{x} = A^+\begin{bmatrix}12\\10\end{bmatrix} = \boxed{\begin{bmatrix}2\\6\\2\end{bmatrix}}$

D) $\hat{x} = 2\begin{bmatrix}1\\2\\-1\end{bmatrix} + 2\begin{bmatrix}0\\1\\2\end{bmatrix}$, so $\hat{x} \in C(A^T)$

② a) If λ is an eigenvalue of A, then λ^4 is an eigenvalue of $A^4 = 0$; so $\lambda^4 = 0$ and therefore $\boxed{\lambda = 0}$ for every eigenvalue of A.

b) If $A^4 = 0$ and A is diagonalizable, there is an invertible matrix S with $S^{-1}AS = D = 0$ by part a); so $A = S0S^{-1} = \boxed{0}$

③ a) $\|Ux\|^2 = \langle Ux, Ux\rangle = \langle x, U^H U x\rangle = \langle x,x\rangle = \|x\|^2$ since $U^H U = I$, so $\|Ux\| = \|x\|$ for any x in $\mathbb{C}^n$

b) Let λ be an eigenvalue of U, so $Ux = \lambda x$ for some $x \neq 0$. Then $\langle Ux, Ux\rangle = \langle \lambda x, \lambda x\rangle = \bar\lambda\langle x, \lambda x\rangle = \bar\lambda\lambda\langle x,x\rangle = |\lambda|^2\langle x,x\rangle$, and $\langle Ux,Ux\rangle = \langle x,x\rangle$ by part a); so $|\lambda|^2 = 1$ since $x \neq 0$ and therefore $|\lambda| = 1$.

④ a) If $x \in \mathbb{C}^n$, $\|PAx\|^2 = \langle PAx, PAx\rangle = \langle Ax, P^H PAx\rangle = \langle Ax, Ax\rangle = \|Ax\|^2$ since $P^H P = I$; so $\|PA\|_2 = \max\limits_{\|x\|=1}\|PAx\| = \max\limits_{\|x\|=1}\|Ax\| = \|A\|_2$

b) If $x \in \mathbb{C}^n$, let $y = Qx \in \mathbb{C}^n$. Since Q is unitary, $\|y\| = \|Qx\| = \|x\|$ as in part a); so $\|AQ\|_2 = \max\limits_{\|x\|=1}\|AQx\| = \max\limits_{\|y\|=1}\|Ay\| = \|A\|_2$

⑤ a) $A = U\Sigma V^T = \boxed{\begin{bmatrix}-1/\sqrt6&-1/\sqrt2&1/\sqrt3\\2/\sqrt6&0&1/\sqrt3\\-1/\sqrt6&1/\sqrt2&1/\sqrt3\end{bmatrix}\begin{bmatrix}\sqrt3&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}1/\sqrt2&-1/\sqrt2\\1/\sqrt2&1/\sqrt2\end{bmatrix}}$

b) $A_1 = \sigma_1 u_1 v_1^T = \boxed{\begin{bmatrix}-1/2&1/2\\1&-1\\-1/2&1/2\end{bmatrix}}$

⑥ a) Since $r = n \le m$, $\Sigma = \begin{bmatrix}\begin{matrix}\sigma_1&&0\\0&\ddots&\sigma_n\end{matrix}\\\hline 0\end{bmatrix} = \begin{bmatrix}D\\\hline 0\end{bmatrix}$ and

$A^*A = V\Sigma^*U^*U\Sigma V^* = V\Sigma^*\Sigma V^* = VD^2V^*$ (where $\underline{A^* = A^H}$)

so $(A^*A)^{-1}A^* = (VD^2V^*)^{-1}A^* = [(V^*)^{-1}(D^2)^{-1}V^{-1}][V\Sigma^*U^*]$

$= V(D^{-1})^2V^*V[D\,|\,0]U^*$

$= V(D^{-1})^2[D\,|\,0]U^*$

$= V[D^{-1}\,|\,0]U^* = V\Sigma^+U^* = A^+$

(7) Let A be a Hermitian matrix, so $A^H = A$; and let λ be an eigenvalue of A, so $Ax = \lambda x$ for some $x \neq 0$.

Then $\langle Ax, x\rangle = \langle \lambda x, x\rangle = \bar{\lambda}\langle x, x\rangle$ and $\langle Ax, x\rangle = \langle x, A^H x\rangle = \langle x, Ax\rangle = \langle x, \lambda x\rangle = \lambda\langle x, x\rangle$,

so $\bar{\lambda}\langle x,x\rangle = \lambda\langle x,x\rangle$ and therefore $\bar{\lambda} = \lambda$ since $\langle x,x\rangle \neq 0$. Thus $\lambda \in \mathbb{R}$.

(8) $A^TAx = A^Tb$ gives $\begin{bmatrix} 14 & 9 \\ 9 & 6 \end{bmatrix} x = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$, so $x = \frac{1}{3}\begin{bmatrix} 6 & -9 \\ -9 & 14 \end{bmatrix}\begin{bmatrix} -1 \\ -3 \end{bmatrix} = \boxed{\begin{bmatrix} 7 \\ -11 \end{bmatrix}}$

(9) a) $\boxed{S = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}}$ gives $S^{-1}AS = \boxed{D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}}$

b) $u = e^{At}u(0) = Se^{Dt}S^{-1}u(0) = 6e^{4t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 5e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so $\boxed{\begin{array}{l} v(t) = 18e^{4t} + 5e^{-t} \\ w(t) = 6e^{4t} + 10e^{-t} \end{array}}$

c) if $u_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$, $u_n = A^n u_0 = SD^nS^{-1}u_0 = 0(4)^n\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 1(-1)^n\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} (-1)^n \\ 2(-1)^n \end{bmatrix}$, so $\boxed{\begin{array}{l} x_n = (-1)^n \\ y_n = 2(-1)^n \end{array}}$

(10) $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 5 \\ -2 & 0 & 0 \end{bmatrix}$

1) $w_1 = v_1 = (1, 2, -2)$

2) $w_2 = v_2 - \left(\frac{v_2 \cdot w_1}{w_1 \cdot w_1}\right) w_1 = (1,1,0) - \frac{3}{9}(1,2,-2) = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ Let $w_2^* = (2,1,2)$

3) $w_3 = v_3 - \left(\frac{v_3 \cdot w_1}{w_1 \cdot w_1}\right) w_1 - \left(\frac{v_3 \cdot w_2^*}{w_2^* \cdot w_2^*}\right) w_2^* = (-1,5,0) - \frac{9}{9}(1,2,-2) - \frac{3}{9}(2,1,2)$

$= (-1,5,0) - (1,2,-2) - \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) = \left(-\frac{8}{3}, \frac{8}{3}, \frac{4}{3}\right) = \frac{4}{3}(-2,2,1)$ Let $w_3^* = (-2,2,1)$

$q_1 = \frac{w_1}{\|w_1\|} = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$ $q_2 = \frac{w_2^*}{\|w_2^*\|} = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ $q_3 = \frac{w_3^*}{\|w_3^*\|} = \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

so $A = QR$ where $\boxed{Q = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}}$ and $R = Q^TA = \boxed{\begin{bmatrix} 3 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}}$

(11) If λ is an eigenvalue of A, $Ax = \lambda x$ for some $x \neq 0$.

Since B is similar to A, $B = S^{-1}AS$ for some invertible matrix S.

Then $A = SBS^{-1}$ gives $(SBS^{-1})x = \lambda x$, so $B(S^{-1}x) = S^{-1}(\lambda x) = \lambda(S^{-1}x)$

where $S^{-1}x \neq 0$ (since otherwise $x = 0$). Therefore λ is an eigenvalue of B.

[or use $B = S^{-1}AS$ to show that $\det(A - \lambda I) = \det(B - \lambda I)$.]

(12) $q(x,y,z) = 5x^2 - 8xy + 3y^2 + 8yz + z^2$

Let $A = \begin{bmatrix} 5 & -4 & 0 \\ -4 & 3 & 4 \\ 0 & 4 & 1 \end{bmatrix}$, so $q(x,y,z) = x^TAx$.

$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -4 & 0 \\ -4 & 3-\lambda & 4 \\ 0 & 4 & 1-\lambda \end{vmatrix} = (5-\lambda)\begin{vmatrix} 3-\lambda & 4 \\ 4 & 1-\lambda \end{vmatrix} - (-4)\begin{vmatrix} -4 & 4 \\ 0 & 1-\lambda \end{vmatrix}$

$= (5-\lambda)(\lambda^2 - 4\lambda - 13) - 16(1-\lambda) = -\lambda^3 + 9\lambda^2 + 9\lambda - 81$

$= \lambda^2(-\lambda + 9) - 9(-\lambda + 9) = (9-\lambda)(\lambda^2 - 9) = (9-\lambda)(\lambda-3)(\lambda+3) = 0$

if $\lambda = 9$, $\lambda = 3$, or $\lambda = -3$

Max. value on $x^2+y^2+z^2 = 1$: $\boxed{9}$

Min. value on $x^2+y^2+z^2 = 1$: $\boxed{-3}$

(13) LET $S = \{v_1, v_2, v_3, v_4\}$ AND $T = \{w_1, w_2, w_3, w_4\}$ BE BASES FOR V AND W, RESPECTIVELY.

THEN $\{v_1, \ldots, v_4, w_1, \ldots, w_4\}$ IS LD (SINCE IT IS A SET OF 8 VECTORS IN $\mathbb{R}^7$),

SO $a_1 v_1 + \cdots + a_4 v_4 + a_5 w_1 + \cdots + a_8 w_4 = 0$ (WITH SOME $a_i \neq 0$).

IF $\underline{x = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4} = \underline{-a_5 w_1 - a_6 w_2 - a_7 w_3 - a_8 w_4}$, THEN $x \in V \cap W$ AND

$x \neq 0$ SINCE $x = 0 \Rightarrow a_i = 0$ FOR $1 \le i \le 4$ AND FOR $5 \le i \le 8$ SINCE S AND T ARE LI,

AND THIS CONTRADICTS THE FACT THAT NOT ALL THE a_i ARE 0.

$\boxed{\text{OR}}$ USING THE RESULT IN #15,

$\dim(V+W) = \dim(V) + \dim(W) - \dim(V \cap W) = 4 + 4 - \dim(V \cap W) = 8 - \dim(V \cap W)$,

AND $\dim(V+W) \le 7$ SINCE $V+W$ IS A SUBSPACE OF $\mathbb{R}^7$; SO

$\dim(V \cap W) = 8 - \dim(V+W) \ge 1$ AND THEREFORE $V \cap W \neq \{\vec{0}\}$.

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(14) $S = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$ GIVES $S^{-1} A S = J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, SO

$$A^n = S J^n S^{-1} = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{bmatrix} \left(\frac{1}{1} \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} \right) = \boxed{2^{n-1} \begin{bmatrix} 2-3n & 9n \\ -n & 2+3n \end{bmatrix}}$$

(15) LET $U = \{x_1, \ldots, x_r\}$ BE A BASIS FOR $V \cap W$, AND EXTEND U TO A BASIS

$S = \{x_1, \ldots, x_r, v_1, \ldots, v_s\}$ FOR V AND A BASIS $T = \{x_1, \ldots, x_r, w_1, \ldots, w_t\}$ FOR W.

THEN $\underline{B = \{x_1, \ldots, x_r, v_1, \ldots, v_s, w_1, \ldots, w_t\} \text{ IS A BASIS FOR } V+W}$, SINCE

1) IF $x \in V+W$, THEN $x = v + w$ WITH $v \in V$ AND $w \in W$; SO

$v = a_1 x_1 + \cdots + a_r x_r + a_{r+1} v_1 + \cdots + a_{r+s} v_s$ AND

$w = b_1 x_1 + \cdots + b_r x_r + b_{r+1} w_1 + \cdots + b_{r+t} w_t$ GIVES

$x = (a_1 + b_1) x_1 + \cdots + (a_r + b_r) x_r + a_{r+1} v_1 + \cdots + a_{r+s} v_s + b_{r+1} w_1 + \cdots + b_{r+t} w_t$.

THEREFORE $\underline{B \text{ SPANS } V+W}$.

2) IF $a_1 x_1 + \cdots + a_r x_r + b_1 v_1 + \cdots + b_s v_s + c_1 w_1 + \cdots + c_t w_t = 0$,

THEN $u = a_1 x_1 + \cdots + a_r x_r + b_1 v_1 + \cdots + b_s v_s = -c_1 w_1 - \cdots - c_t w_t \in V \cap W$;

SO $u = d_1 x_1 + \cdots + d_r x_r$ FOR SOME SCALARS d_i AND THEREFORE

$a_1 x_1 + \cdots + a_r x_r + b_1 v_1 + \cdots + b_s v_s = d_1 x_1 + \cdots + d_r x_r$,

THEN $a_i = d_i$ AND $\underline{b_j = 0}$ FOR $1 \le i \le r$ AND $1 \le j \le s$ SINCE S IS A BASIS FOR V,
(AND THEREFORE IS LI)

SO $\underline{a_1 x_1 + \cdots + a_r x_r + c_1 w_1 + \cdots + c_t w_t = 0}$,

THEN $\underline{a_1 = 0, \ldots, a_r = 0}$ AND $\underline{c_1 = 0, \ldots, c_t = 0}$ SINCE T IS LI.

THEREFORE $\underline{B \text{ IS LI}}$.

THEN $\underline{\dim(V+W)} = r + s + t = (r+s) + (r+t) - r$

$= \underline{\dim(V) + \dim(W) - \dim(V \cap W)}$.

(WHEN V AND W ARE SUBSPACES OF $\mathbb{R}^n$, THIS CAN BE SHOWN USING MATRICES; SEE EX. 6 ON P. 416.)