MATH 17B

SOLUTIONS – PINK SHEET

① $L = \begin{bmatrix} 3/2 & 4 \\ 1/4 & 0 \end{bmatrix}$

a) $\det(L-\lambda I) = \begin{vmatrix} \frac{3}{2}-\lambda & 4 \\ \frac{1}{4} & -\lambda \end{vmatrix} = \left(\frac{3}{2}-\lambda\right)(-\lambda) - 1 = \lambda^2 - \frac{3}{2}\lambda - 1 = 0$

$2\lambda^2 - 3\lambda - 2 = 0 \qquad (2\lambda+1)(\lambda-2) = 0 \qquad \boxed{\lambda = -\frac{1}{2}}$ or $\boxed{\lambda = 2}$

b) $\boxed{\lambda_1 = 2}$ is the long-term growth rate.

c) $(L-2I)x = 0$ gives $\begin{bmatrix} -\frac{1}{2} & 4 & 0 \\ \frac{1}{4} & -2 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 1 & -8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ Let $y = t$, so $x = 8t$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8t \\ t \end{bmatrix} = t\begin{bmatrix} 8 \\ 1 \end{bmatrix}$, so $v_1 = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 2$

Long-term age distribution: $\begin{bmatrix} 8/9 \\ 1/9 \end{bmatrix} \approx \boxed{\begin{bmatrix} 88.9\% \\ 11.1\% \end{bmatrix}}$

② $L = \begin{bmatrix} 1/2 & 3/8 \\ 1/2 & 0 \end{bmatrix}$

a) $\det(L-\lambda I) = \begin{vmatrix} \frac{1}{2}-\lambda & 3/8 \\ 1/2 & -\lambda \end{vmatrix} = \left(\frac{1}{2}-\lambda\right)(-\lambda) - \frac{3}{16} = \lambda^2 - \frac{1}{2}\lambda - \frac{3}{16} = 0$

$16\lambda^2 - 8\lambda - 3 = 0 \qquad (4\lambda+1)(4\lambda-3) = 0 \qquad \boxed{\lambda = -\frac{1}{4}}$ or $\boxed{\lambda = \frac{3}{4}}$

b) $\boxed{\lambda_1 = \frac{3}{4}}$ is the long-term growth rate

c) $(L - \frac{3}{4}I)x = 0$ gives $\begin{bmatrix} -1/4 & 3/8 & 0 \\ 1/2 & -3/4 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ Let $y = 2t$, so $x = 3t$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3t \\ 2t \end{bmatrix} = t\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, so $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda_1 = \frac{3}{4}$

Long-term age distribution: $\begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix} = \boxed{\begin{bmatrix} 60\% \\ 40\% \end{bmatrix}}$

③ $L = \begin{bmatrix} 2/3 & 1 \\ 1/3 & 0 \end{bmatrix}$

a) $\det(L-\lambda I) = \begin{vmatrix} 2/3-\lambda & 1 \\ 1/3 & -\lambda \end{vmatrix} = (2/3-\lambda)(-\lambda) - \frac{1}{3} = \lambda^2 - \frac{2}{3}\lambda - \frac{1}{3} = 0$

$3\lambda^2 - 2\lambda - 1 = 0 \qquad (3\lambda+1)(\lambda-1) = 0 \qquad \boxed{\lambda = -\frac{1}{3}}$ or $\boxed{\lambda = 1}$

b) $\boxed{\lambda_1 = 1}$ is the long-term growth rate

c) $(L-I)x = 0$ gives $\begin{bmatrix} -1/3 & 1 & 0 \\ 1/3 & -1 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ Let $y = t$, so $x = 3t$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, so $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 1$.

Long-term age distribution: $\begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} = \boxed{\begin{bmatrix} 75\% \\ 25\% \end{bmatrix}}$

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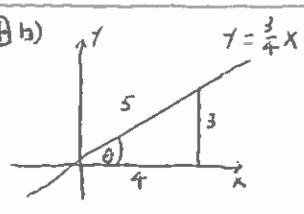
④ b)



$\tan\theta = m = \frac{3}{4}$, so $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$

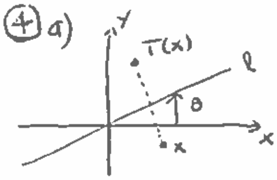
Then $\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$

and $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$, so

$A = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \boxed{\begin{bmatrix} 7/25 & 24/25 \\ 24/25 & -7/25 \end{bmatrix}}$ is the matrix for the linear map $T: \mathbb{R}^2 \to \mathbb{R}^2$ which reflects each point in the line $y = \frac{3}{4}x$.

MATH 17B SOLUTIONS — PINK SHEET AND CH. 9 RE

P.S. — 4 a)



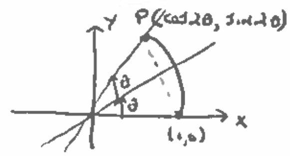
$$A = A_3 A_2 A_1 = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{(ROTATION OF } \theta)} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\text{(REFLECTION IN X-AXIS)}} \underbrace{\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}}_{\text{(ROTATION OF } -\theta)}$$

$$\text{so } A = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}}_{A_3 A_2} \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{A_1} = \boxed{\begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \end{bmatrix}}$$

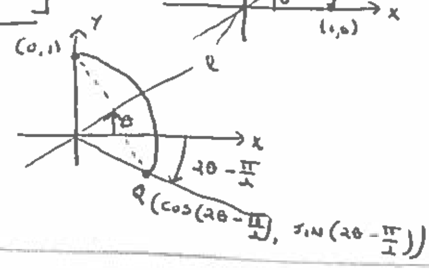
$$= \boxed{\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}}$$

REMARK WE COULD ALSO GET THIS USING

$$A = \left[T(e_1) \,\middle|\, T(e_2) \right], \quad \text{WHERE } T(e_1) = \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}$$



$$\text{AND } T(e_2) = \begin{bmatrix} \cos(2\theta - \frac{\pi}{2}) \\ \sin(2\theta - \frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} \sin 2\theta \\ -\cos 2\theta \end{bmatrix}$$



(See The Previous PAGE For 4 b.)

CH. 9 RE — 3

$$L = \begin{bmatrix} 3/2 & 7/8 \\ 1/2 & 0 \end{bmatrix} \qquad \det(L - \lambda I) = \begin{vmatrix} 3/2 - \lambda & 7/8 \\ 1/2 & -\lambda \end{vmatrix} = \left(\tfrac{3}{2} - \lambda\right)(-\lambda) - \tfrac{7}{16} = \lambda^2 - \tfrac{3}{2}\lambda - \tfrac{7}{16} = 0$$

$16\lambda^2 - 24\lambda - 7 = 0 \qquad (4\lambda - 7)(4\lambda + 1) = 0 \qquad \lambda = 7/4$ OR $\lambda = -1/4$

so $\boxed{\lambda_1 = \frac{7}{4}}$ IS THE LONG-TERM GROWTH RATE

$$(L - \tfrac{7}{4} I)x = 0 \quad \text{GIVES} \quad \begin{bmatrix} -1/4 & 7/8 & 0 \\ 1/2 & -7/4 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 1 & -7/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

LET $y = 2t$, SO $x = 7t$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7t \\ 2t \end{bmatrix} = t\begin{bmatrix} 7 \\ 2 \end{bmatrix}, \quad \text{SO } v_1 = \boxed{\begin{bmatrix} 7 \\ 2 \end{bmatrix}} \text{ IS A STABLE AGE DISTRIBUTION}$$

REMARK ANY EIGENVECTOR (WITH POSITIVE COMPONENTS) FOR λ_1 WILL GIVE A STABLE AGE DISTRIBUTION, AND

THE LONG-TERM AGE DISTRIBUTION IS $\begin{bmatrix} 7/9 \\ 2/9 \end{bmatrix}$.

11

$$L = \begin{bmatrix} .5 & 2.3 \\ a & 0 \end{bmatrix} \qquad \det(L - \lambda I) = \begin{vmatrix} .5 - \lambda & 2.3 \\ a & -\lambda \end{vmatrix} = (.5 - \lambda)(-\lambda) - 2.3a$$

$$= \lambda^2 - .5\lambda - 2.3a = 0 \quad \text{IF}$$

$$\lambda = \frac{.5 \pm \sqrt{(.5)^2 - 4(-2.3a)}}{2} = \frac{.5 \pm \sqrt{.25 + 9.2a}}{2},$$

so $\lambda_1 > 1$ IFF $\dfrac{.5 + \sqrt{.25 + 9.2a}}{2} > 1$ IFF $.5 + \sqrt{.25 + 9.2a} > 2$ IFF

$\sqrt{.25 + 9.2a} > \frac{3}{2}$ IFF $.25 + 9.2a > 2.25$ IFF $9.2a > 2$ IFF $a > \dfrac{2}{9.2} = \dfrac{20}{92} = \dfrac{5}{23}$,

so THE POPULATION HAS LONG-TERM GROWTH FOR a IN $\boxed{\left(\frac{5}{23}, 1\right]}$

(SINCE $0 \le a \le 1$)