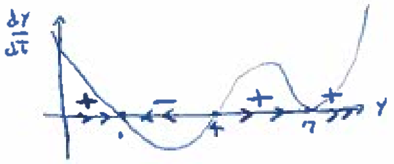
① $\dfrac{dy}{dt} = 5(y-1)(y-4)(y-7)^2$



$\boxed{\begin{array}{l} y=1: \text{ stable} \\ y=4: \text{ unstable} \\ y=7: \text{ semistable} \end{array}}$

② $x + 2y + 2z + 4w = 3$
$2x + 3y + z + 2w = 7$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 4 & 3 \\ 2 & 3 & 1 & 2 & 7 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 4 & 3 \\ 0 & -1 & -3 & -6 & 1 \end{array}\right] \xrightarrow[-R_2]{2R_2+R_1} \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & -4 & -8 & 5 \\ 0 & 1 & 3 & 6 & -1 \end{array}\right] \end{array}$$

Let $z = t$, $w = u$:

$\boxed{x = 5 + 4t + 8u, \quad y = -1 - 3t - 6u, \quad z = t, \quad w = u}$ where t, u are in $\mathbb{R}$.

③ $xy' - 2y = 8x^4 - 5x^2$; $(1, 7)$

1) $y' - \frac{2}{x}y = 8x^3 - 5x$ 2) $u(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = (e^{\ln x})^{-2} = x^{-2}$

3) $x^{-2}\left[y' - \frac{2}{x}y\right] = x^{-2}\left[8x^3 - 5x\right]$ so $(x^{-2}y)' = 8x - 5x^{-1}$

4) $x^{-2}y = \int (8x - 5x^{-1})\,dx = 4x^2 - 5\ln x + C$ so

$y = 4x^4 - 5x^2 \ln x + Cx^2$

When $x = 1$, $y = 7$: $4 - 0 + C = 7$ so $C = 3$: $\boxed{y = 4x^4 - 5x^2 \ln x + 3x^2}$

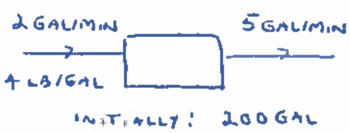
REMARK: If the given point had a negative x-coordinate, or no point were given, we would use $\ln|x|$ instead of $\ln x$.

④ $\dfrac{dy}{dt} = t^2(y^2 + 9)$ so $\displaystyle\int \frac{1}{y^2+9}\,dy = \int t^2\,dt$, $\frac{1}{3}\tan^{-1}\frac{y}{3} = \frac{1}{3}t^3 + C$,

$\tan^{-1}\frac{y}{3} = t^3 + D$, $\frac{y}{3} = \tan(t^3 + D)$, $y = 3\tan(t^3 + D)$

If $t = 0$, $y = 3$; so $\tan^{-1} 1 = D$ and $D = \frac{\pi}{4}$: $\boxed{y = 3\tan\left(t^3 + \frac{\pi}{4}\right)}$

⑤



$\boxed{\dfrac{dA}{dt} = 2(4) - 5\left(\dfrac{A}{200 - 3t}\right)}$

Net loss: 3 gal/min

⑥ $N(t) = \dfrac{600}{1 + ae^{-rt}}$

1) If $t = 0$, $N = 100$: $100 = \dfrac{600}{1+a}$, $1 + a = 6$, $a = 5$ so

$N(t) = \dfrac{600}{1 + 5e^{-rt}}$

2) If $t = 15$, $N = 200$: $200 = \dfrac{600}{1 + 5e^{-15r}}$, $1 + 5e^{-15r} = 3$,

$5e^{-15r} = 2$, $e^{-15r} = \frac{2}{5}$, $e^{-r} = \left(\frac{2}{5}\right)^{\frac{1}{15}}$ so $N(t) = \dfrac{600}{1 + 5(e^{-r})^t} = \dfrac{600}{1 + 5\left(\frac{2}{5}\right)^{t/15}}$.

3) If $N = 400$, $\dfrac{600}{1 + 5\left(\frac{2}{5}\right)^{t/15}} = 400$, $\frac{3}{2} = 1 + 5\left(\frac{2}{5}\right)^{t/15}$, $\frac{1}{2} = 5\left(\frac{2}{5}\right)^{t/15}$,

$\left(\frac{2}{5}\right)^{t/15} = \frac{1}{10}$, $\frac{t}{15}\ln\frac{2}{5} = \ln\frac{1}{10}$, $t = \boxed{\dfrac{15\ln .1}{\ln .4}\text{ yr}} = \boxed{\dfrac{15\ln 10}{\ln 2.5}\text{ yr}} = \boxed{\dfrac{15\ln\frac{1}{10}}{\ln\frac{2}{5}}\text{ yr}}$

(7) $A = \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 5 & 4 \\ 6 & 8 \end{bmatrix}$

$$(AB^T)^{-1} = (B^T)^{-1}A^{-1} = (B^{-1})^T A^{-1} = \begin{bmatrix} 5 & 6 \\ 4 & 8 \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}\right) = \begin{bmatrix} 5 & 6 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -3/2 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} -4 & 7 \\ -8 & 12 \end{bmatrix}}$$

(8) $A = \begin{bmatrix} 4 & 1 \\ 8 & -3 \end{bmatrix}$ $\quad \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ 8 & -3-\lambda \end{vmatrix} = (4-\lambda)(-3-\lambda) - 8 = \lambda^2 - \lambda - 20 = (\lambda - 5)(\lambda + 4) = 0$

if $\boxed{\lambda = 5}$ or $\boxed{\lambda = -4}$

a) $\lambda = 5$: $(A - 5I)x = 0$ gives $\begin{bmatrix} -1 & 1 & 0 \\ 8 & -8 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{matrix} x & y & \end{matrix} \\ \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ let $y = t$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so $\boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ is an eigenvector for $\lambda = 5$

b) $\lambda = -4$: $(A - (-4)I)x = 0$ gives $\begin{bmatrix} 8 & 1 & 0 \\ 8 & 1 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{matrix} x & y & \end{matrix} \\ \begin{bmatrix} 1 & 1/8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ let $y = 8t$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ 8t \end{bmatrix} = t\begin{bmatrix} -1 \\ 8 \end{bmatrix}$, so $\boxed{\begin{bmatrix} -1 \\ 8 \end{bmatrix}}$ is an eigenvector for $\lambda = -4$

(9) $N(1) = LN(0) = \begin{bmatrix} 1.5 & 5 & 3.2 \\ .5 & 0 & 0 \\ 0 & .8 & 0 \end{bmatrix}\begin{bmatrix} 40 \\ 30 \\ 50 \end{bmatrix} = \boxed{\begin{bmatrix} 370 \\ 20 \\ 24 \end{bmatrix}}$

(10) $\frac{dy}{dx} = 8x^3y - 2x^3y^2 = 2x^3y(4-y)$, $\quad \int \frac{1}{y(4-y)}\,dy = \int 2x^3\,dx$,

$\frac{1}{y(4-y)} = \frac{A}{y} + \frac{B}{4-y}$

$1 = A(4-y) + By$

$y=0$: $1 = 4A$ $\quad A = 1/4$

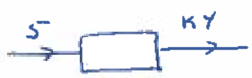
$y=4$: $1 = 4B$ $\quad B = 1/4$

$\int\left(\frac{1/4}{y} + \frac{1/4}{4-y}\right)dy = \frac{1}{2}x^4 + C$, $\quad \frac{1}{4}\int\left(\frac{1}{y} - \frac{-1}{4-y}\right)dy = \frac{1}{2}x^4 + C$,

$\ln y - \ln(4-y) = 2x^4 + D$, $\quad \ln\left(\frac{y}{4-y}\right) = 2x^4 + D$,

$\frac{y}{4-y} = e^{2x^4 + D} = Ae^{2x^4}$, $\quad \frac{4-y}{y} = \frac{1}{Ae^{2x^4}} = ae^{-2x^4}$, $\quad \frac{4}{y} - 1 = ae^{-2x^4}$,

$\frac{4}{y} = 1 + ae^{-2x^4}$, $\quad \frac{y}{4} = \frac{1}{1 + ae^{-2x^4}}$, $\quad \boxed{y = \frac{4}{1 + ae^{-2x^4}}}$

(11) 

$\frac{dy}{dt} = 5 - ky$

1) $\frac{dy}{dt} + ky = 5$

2) $u(t) = e^{\int k\,dt} = e^{kt}$

3) $e^{kt}\left[\frac{dy}{dt} + ky\right] = 5e^{kt}$

$(e^{kt}y)' = 5e^{kt}$

4) $e^{kt}y = \int 5e^{kt}\,dt = \frac{5}{k}e^{kt} + C$

$y = \frac{5}{k} + Ce^{-kt}$

OR

$\int \frac{1}{5 - ky}\,dy = \int dt$

$-\frac{1}{k}\int \frac{-k}{5 - ky}\,dy = t + C$

$-\frac{1}{k}\ln(5 - ky) = t + C$

$\ln(5 - ky) = -kt + D$

$5 - ky = e^{-kt + D} = Ae^{-kt}$

$ky = 5 - Ae^{-kt}$

$y = \frac{5}{k} - Be^{-kt}$

A) if $t = 0$, $y = 10$; so $\frac{5}{k} + C = 10$, $\quad C = 10 - \frac{5}{k}$

so $y = \frac{5}{k} + \left(10 - \frac{5}{k}\right)e^{-kt}$

B) $\lim_{t \to \infty} y = \lim_{t \to \infty}\left[\frac{5}{k} + \left(10 - \frac{5}{k}\right)e^{-kt}\right] = \frac{5}{k} + 0 = \frac{5}{k} = 100$, $\quad$ so $k = .05 = \frac{1}{20}$

so $\boxed{y = 100 - 90e^{-.05t}}$

or $\boxed{y = 100 - 90e^{-t/20}}$