(20) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{2^n}$

A) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n!}{2^n}$ $\quad \rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)!}{2^{n+1}} \cdot \frac{2^n}{n!} = \lim_{n\to\infty} \frac{n+1}{2} = \infty > 1,$

so $\sum_{n=1}^{\infty} |a_n|$ DIVERGES BY THE RATIO TEST AND THEREFORE $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{DIVERGES}}$ ALSO.

(22) $\sum_{n=1}^{\infty} (-1)^n \frac{\sin n}{n^2}$

A) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$ COMPARE TO $\sum_{n=1}^{\infty} \frac{1}{n^2}$, WHICH CONVERGES SINCE IT IS A P-SERIES WITH $P > 1$:

$\frac{|\sin n|}{n^2} \le \frac{1}{n^2}$ FOR ALL n, SO $\sum_{n=1}^{\infty} |a_n|$ CONVERGES BY THE COMPARISON TEST

AND THEREFORE $\sum_{n=1}^{\infty} (-1)^n \frac{\sin n}{n^2}$ $\boxed{\text{CONVERGES ABSOLUTELY}}$.

(24) $\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n+5^n}$

A) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{2^{n+1}}{n+5^n}$ COMPARE TO $\sum_{n=1}^{\infty} \frac{2^{n+1}}{5^n} = \sum_{n=1}^{\infty} 2\left(\frac{2}{5}\right)^n$, WHICH CONVERGES SINCE IT IS A GEOMETRIC SERIES WITH $|r| < 1$:

$\frac{2^{n+1}}{n+5^n} \le \frac{2^{n+1}}{5^n}$ FOR ALL n, SO $\sum_{n=1}^{\infty} |a_n|$ CONVERGES BY THE COMPARISON TEST

AND THEREFORE $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{CONVERGES ABSOLUTELY}}$

(OR USE THE LCT AND THE FACT THAT $\lim_{n\to\infty} \frac{2^{n+1}}{n+5^n} \div \frac{2^{n+1}}{5^n} = \lim_{n\to\infty} \frac{1}{\frac{n}{5^n}+1} = \frac{1}{0+1} = 1 \ne \infty$

SINCE $\lim_{n\to\infty} \frac{n}{5^n} = \lim_{x\to\infty} \frac{x}{5^x} \overset{(\frac{\infty}{\infty})}{=} \lim_{x\to\infty} \frac{1}{5^x \ln 5} = 0$,

OR USE THE RATIO TEST AND THE FACT THAT
$\lim_{n\to\infty} \frac{2^{n+2}}{n+1+5^{n+1}} \cdot \frac{n+5^n}{2^{n+1}} = \lim_{n\to\infty} 2 \cdot \frac{n+5^n}{n+1+5^{n+1}} = \lim_{n\to\infty} 2 \cdot \frac{\frac{n}{5^n}+1}{\frac{n}{5^n}+\frac{1}{5^n}+5} = 2 \cdot \frac{0+1}{0+0+5} = \frac{2}{5} < 1$

SINCE $\lim_{n\to\infty} \frac{n}{5^n} = 0$ AS ABOVE.)

(21) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+3}$

A) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n+3}$ DIVERGES, SINCE IT IS THE HARMONIC SERIES WITH 3 TERMS DELETED.

B) SINCE 1) $\lim_{n\to\infty} \frac{1}{n+3} = 0$ AND 2) $\frac{1}{n+3} \ge \frac{1}{n+4}$ FOR ALL n,

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+3}$ CONVERGES BY THE AST; SO IT $\boxed{\text{CONVERGES CONDITIONALLY}}$.

10.6 – (40) $\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^2 3^n}{(2n+1)!}$

A) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{(n!)^2 3^n}{(2n+1)!}$

$$\ell = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{((n+1)!)^2 3^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{(n!)^2 3^n} = \lim_{n\to\infty} \frac{3(n+1)^{\cancel{2}}}{(2n+3)(2n+2)}$$

$$= \lim_{n\to\infty} \frac{3n+3}{4n+6} = \frac{3}{4} < 1,$$ so $\sum_{n=1}^{\infty} |a_n|$ CONVERGES BY THE RATIO TEST AND THEREFORE $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{CONVERGES ABSOLUTELY}}$.

TAN – (2) $\sum_{n=2}^{\infty} (-1)^n \left(1 - \frac{1}{3n}\right)^n$

$$\lim_{n\to\infty} \left(1 - \frac{1}{3n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{-1/3}{n}\right)^n = e^{-1/3} \neq 0,$$ so

THE SERIES $\boxed{\text{DIVERGES}}$ BY THE DIVERGENCE TEST.

(3) $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{3^n n!}$

A) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n^n}{3^n n!}$

$$\ell = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{3^{n+1}(n+1)!} \cdot \frac{3^n n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{3(n+1)n^n} = \lim_{n\to\infty} \frac{(n+1)^n}{3n^n}$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^n = \frac{1}{3}\cdot e = \frac{e}{3} < 1,$$ so $\sum_{n=1}^{\infty} |a_n|$ CONVERGES BY THE RATIO TEST AND THEREFORE $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{CONVERGES ABSOLUTELY}}$.

(4) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln n)^2}$

A) $\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ $\quad f(x) = \frac{1}{x(\ln x)^2}$ IS CONTINUOUS AND DECREASING FOR $x \geq 2$, AND

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{T\to\infty} \int_2^T \frac{1}{x(\ln x)^2}\,dx = \lim_{T\to\infty} \left[-\frac{1}{\ln x}\right]_2^T \quad (u = \ln x,\ du = \tfrac{1}{x}\,dx)$$

$$= \lim_{T\to\infty} \left(-\frac{1}{\ln T} - \left(-\frac{1}{\ln 2}\right)\right) = \frac{1}{\ln 2},$$ so $\sum_{n=2}^{\infty} |a_n|$ CONVERGES BY THE INTEGRAL TEST AND THEREFORE $\sum_{n=2}^{\infty} a_n$ $\boxed{\text{CONVERGES ABSOLUTELY}}$.

(9) $\sum_{n=1}^{\infty} (-1)^n \frac{n^{n+1}}{(2n-1)^n}$

A) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n^{n+1}}{(2n-1)^n}$

$$\ell = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{n^{n+1}}{(2n-1)^n}\right)^{1/n} = \lim_{n\to\infty} \frac{n^{1+\frac{1}{n}}}{2n-1} = \lim_{n\to\infty} \frac{n(n^{1/n})}{2n-1}$$

$$= \lim_{n\to\infty} \left(\frac{n}{2n-1}\right) \cdot \lim_{n\to\infty} n^{1/n} = \frac{1}{2}\cdot 1 = \frac{1}{2} < 1,$$

so $\sum_{n=1}^{\infty} |a_n|$ CONVERGES BY THE ROOT TEST AND THEREFORE

$\sum_{n=1}^{\infty} a_n$ $\boxed{\text{CONVERGES ABSOLUTELY}}$.

MATH 21C SOLUTIONS – TAN SHEET (SEC. 10.6)

⑤ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt[n]{n}}$ $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = \frac{1}{1} = 1 \neq 0$, so the series $\boxed{\text{DIVERGES}}$ by the Divergence Test.

⑦ $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n(n+1)}}$

A) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$ compare to $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges (harmonic series):

$\lim_{n\to\infty} \frac{1}{\sqrt{n(n+1)}} \div \frac{1}{n} = \lim_{n\to\infty} \frac{n}{\sqrt{n}\sqrt{n+1}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \sqrt{1} = 1 \neq 0,$

so $\sum_{n=1}^{\infty} |a_n|$ diverges by the LCT.

B) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n(n+1)}}$ converges by the AST since 1) $\lim_{n\to\infty} \frac{1}{\sqrt{n(n+1)}} = 0$ and 2) $\frac{1}{\sqrt{n(n+1)}} \geq \frac{1}{\sqrt{(n+1)(n+2)}}$ for $n \geq 1$,

so the series $\boxed{\text{CONVERGES CONDITIONALLY}}$.

⑧ $\sum_{n=2}^{\infty} \frac{\cos n}{n(n-1)}$

A) $\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{|\cos n|}{n(n-1)}$ $\frac{|\cos n|}{n(n-1)} \leq \frac{1}{n(n-1)} \leq \frac{2}{n^2}$ for $n \geq 2$ since

$|\cos n| \leq 1$ and $\frac{1}{n(n-1)} \leq \frac{2}{n^2}$ iff $n^2 \leq 2n(n-1)$ iff $n \leq 2(n-1)$ iff $n \leq 2n-2$ iff $n \geq 2$,

and $\sum_{n=2}^{\infty} \frac{2}{n^2}$ converges since it is a multiple of a p-series with $p > 1$ (with 1 term omitted)

so $\sum_{n=2}^{\infty} |a_n|$ converges by the Comparison Test and therefore $\sum_{n=2}^{\infty} \frac{\cos n}{n(n-1)}$ $\boxed{\text{CONVERGES ABSOLUTELY}}$.

(or use the LCT to compare $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ with $\sum_{n=2}^{\infty} \frac{1}{n^2}$.)

⑩ $\sum_{n=1}^{\infty} (-1)^n \frac{5^n (n!)^2}{(2n+1)!}$ A) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{5^n (n!)^2}{(2n+1)!}$ using the Ratio Test,

$\ell = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{5^{n+1} ((n+1)!)^2}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{5^n (n!)^2} = \lim_{n\to\infty} 5 \cdot \left(\frac{(n+1)!}{n!}\right)^2 \cdot \frac{(2n+1)!}{(2n+3)!}$

$= \lim_{n\to\infty} 5(n+1)^2 \cdot \frac{1}{(2n+3)(2n+2)} = \lim_{n\to\infty} \frac{5(n+1)^{\cancel{2}}}{2(2n+3)\cancel{(n+1)}} = \lim_{n\to\infty} \frac{5}{2} \cdot \frac{n+1}{2n+3} = \frac{5}{2} \cdot \frac{1}{2} = \frac{5}{4} > 1,$

so $\sum_{n=1}^{\infty} |a_n|$ diverges by the Ratio Test and therefore $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{DIVERGES}}$ also (since $\lim_{n\to\infty} a_n \neq 0$).

⑪ $\sum_{n=1}^{\infty} (-1)^{n+1} n \sin \frac{4}{n}$ $\lim_{n\to\infty} n \sin \frac{4}{n} = \lim_{n\to\infty} \frac{\sin \frac{4}{n}}{\frac{1}{n}} \overset{\boxed{T=\frac{1}{n}}}{=} \lim_{T\to 0^+} \frac{\sin 4T}{T} \underset{(\frac{0}{0})}{=} \lim_{T\to 0^+} \frac{\cos 4T \cdot 4}{1} = 4 \neq 0,$

so the series $\boxed{\text{DIVERGES}}$ by the Divergence Test.

⑫ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{(n+8)^2}$

A) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{(n+8)^2}$ compare to $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges (harmonic series):

$\lim_{n\to\infty} \frac{n}{(n+8)^2} \div \frac{1}{n} = \lim_{n\to\infty} \frac{n^2}{(n+8)^2} = \lim_{n\to\infty} \left(\frac{n}{n+8}\right)^2 = 1^2 = 1 \neq 0$, so $\sum_{n=1}^{\infty} |a_n|$ diverges by the LCT.

B) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{(n+8)^2}$ converges by the AST since

1) $\lim_{n\to\infty} \frac{n}{(n+8)^2} = \lim_{n\to\infty} \frac{\frac{n}{n^2}}{\frac{(n+8)^2}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\left(\frac{n+8}{n}\right)^2} = \frac{0}{1} = 0$ and (continued on next page)

(or use the degree of the top is less than the degree of the bottom)

(12) (CONTINUED)

2) IF $f(x) = \frac{x}{(x+8)^2}$, THEN $f'(x) = \frac{(x+8)^2 \cdot 1 - x \cdot 2(x+8)}{(x+8)^4} = \frac{8-x}{(x+8)^3} < 0$ FOR $x > 8$;

SO f IS DECREASING FOR $x \geq 8$ AND THEREFORE $\underline{\frac{n}{(n+8)^2} \geq \frac{n+1}{(n+9)^2}}$ FOR $n \geq 8$.

[OR USE $\frac{n}{(n+8)^2} \geq \frac{n+1}{(n+9)^2}$ IFF $n(n^2+18n+81) \geq (n+1)(n^2+16n+64)$ IFF

$n^3+18n^2+81n \geq n^3+17n^2+80n+64$ IFF $n^2+n \geq 64$ IFF $n(n+1) \geq 64$ IFF $\underline{n \geq 8}$]

THEREFORE $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{(n+8)^2}$ $\boxed{\text{CONVERGES CONDITIONALLY}}$ USING THE AST.

(13) $\sum_{n=1}^{\infty} \frac{5^n \sin n}{n!}$

A) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{5^n |\sin n|}{n!}$

$\sum_{n=1}^{\infty} \frac{5^n}{n!}$ CONVERGES BY THE RATIO TEST SINCE

$\ell = \lim_{n \to \infty} \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \lim_{n \to \infty} \frac{5}{n+1} = 0 < 1$;

AND $\frac{5^n |\sin n|}{n!} \leq \frac{5^n}{n!}$ FOR ALL n (SINCE $|\sin n| \leq 1$),

SO $\sum_{n=1}^{\infty} |a_n|$ CONVERGES BY THE COMPARISON TEST AND

THEREFORE $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{CONVERGES ABSOLUTELY}}$.

(14) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n^2}$

A) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ COMPARE TO $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, WHICH CONVERGES SINCE IT IS A P-SERIES WITH $p > 1$;

$\ln n < \sqrt{n}$ FOR $n \geq N$ (FOR SOME N),*

SO $\underline{\frac{\ln n}{n^2} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}}$ FOR $n \geq N$.

THEREFORE $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ CONVERGES BY THE COMPARISON TEST,

SO $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{CONVERGES ABSOLUTELY}}$.

* REMARK IT IS NOT HARD TO SHOW THAT $\ln n < \sqrt{n}$ FOR ALL $n \geq 1$, BY FINDING THE MINIMUM OF $f(x) = \sqrt{x} - \ln x$ FOR $x > 0$.