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Nondecomposable solutions to group equations and an application to polyhedral combinatorics

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Abstract This paper is based on the study of the set of nondecomposable integer solutions in a Gomory corner polyhedron, which was recently used in a reformulation method for integer linear programs. In this paper, we present an algorithm for efficiently computing this set. We precompute a database of nondecomposable solutions for cyclic groups up to order 52. As a second application of this database, we introduce an algorithm for computing nontrivial simultaneous lifting coefficients. The lifting coefficients are exact for a discrete relaxation of the integer program that consists of a group relaxation plus bound constraints.

Keywords: Gomory corner polyhedron – irreducible group solutions – simultaneous lifting

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1 Introduction

At the end of the 1960s, Ralph Gomory proposed an algebraic approach for solving integer programs via the use of group relaxations (Gomory, 1969). The key object which is underlying this approach is the so-called Gomory corner polyhedron associated with an integer programming problem,

$$\begin{aligned} \max\ & c^\top x \\ \text{s.t.}\ & Ax = b \\ & x \in \mathbb{Z}^n_+. \end{aligned} \tag{1}$$

Given a selection of linearly independent columns of A forming the square matrix A_B, we can relax the set of feasible integral points of the IP problem to the set

$$S = \{ x \in \mathbb{Z}^n : A_B x_B + A_{\bar{B}} x_{\bar{B}} = b,\ x_{\bar{B}} \geq 0 \},$$

i.e., the nonnegativity constraints for the variables x_B are dropped while integrality is maintained both for the set B and its complementary set $\bar{B}$. The optimization problem over the set S is called a *group relaxation* of the IP. Since the values of x_B are uniquely determined by the vector $x_{\bar{B}}$, the set S can be bijectively mapped to the set

$$S' = \{x \in \mathbb{Z}_+^{|\bar{B}|} : A_{\bar{B}} x_{\bar{B}} \equiv b \pmod{L(A_B)}\},$$

where $L(A_B) = \{ A_B z : z \in \mathbb{Z}^{|B|} \}$ denotes the lattice generated by the column vectors of A_B.

The convex hull of the set S' is called the Gomory corner polyhedron (with respect to the index set B). By studying the abelian group $\mathbb{Z}^{|B|}/L(A_B)$ and homomorphic images or cyclic subgroups of it, Gomory et al. provided insight into the facet structure of the corner polyhedron. For instance, in his original paper, Gomory (1969) gave a complete description of the facets of corner polyhedra resulting from small cyclic groups. In fact, the investigations reveal many interesting connections between the geometrical structures of an IP problem and corresponding corner polyhedra. Despite the efforts made in examining the corner polyhedron in the 1970s (Gomory and Johnson, 1972a,b) and the ones in recent years (Aráoz et al., 2003; Gomory et al., 2003; Gomory and Johnson, 2003), we are still far away from turning this theoretical knowledge into an algorithmic tool for solving large IP instances.

Recently, Köppe et al. (2004) introduced reformulations for an IP problem based on the corner polyhedron using so-called *irreducible* or *nondecomposable solutions* to a group relaxation and presented possible algorithms based on these ideas. We will briefly summarize their approach.

Let us consider a set of the form

$$S(d) = \{ x \in \mathbb{Z}_+^n : Bx \equiv d \pmod{L(\Delta)} \},$$

where $\Delta \in \mathbb{Z}^{r\times r}$ is a regular diagonal positive integer matrix, and $B \in \mathbb{Z}^{r\times n}$, and $d \in \mathbb{Z}_+^r$. In order to describe a reformulation of $S(d)$, we need to introduce the notion of irreducible solutions.

Definition 1. A non-zero vector $x \in \mathbb{Z}_+^n$ is an *irreducible solution* of $S(d)$ if $x \in S(d)$, and there is no other distinct nonzero $\tilde{x} \in S(d)$ with $\tilde{x} \leq x$. Every irreducible vector x in $S(0)$ is called *homogeneous*. An irreducible vector in $S(d)$ is called *inhomogeneous* whenever $d \neq 0$.

The reformulation is based on the fact that every point in $S(d)$ can be written as the sum of exactly one inhomogeneous irreducible solution of $S(d)$ and a nonnegative integer combination of homogeneous irreducible solutions of $S(0)$. If we collect the inhomogeneous vectors in the matrix C and the homogeneous vectors in the matrix D, we can write $S(d)$ as

$$S(d) = \{x \in \mathbb{Z}_+^n : x = C\lambda + D\mu,\ 1^\top \lambda = 1,\ \lambda \in \mathbb{Z}_+^s,\ \mu \in \mathbb{Z}_+^t\}.$$

Example 2. Consider the set $X = \{(x_1, x_2, x_3) \in \mathbb{Z}_+^3 : 3x_1 + 7x_2 + 9x_3 = 22\}$. By taking the equation modulo 4, we obtain the valid group relaxation

$$S(2) = \{(x_1, x_2, x_3) \in \mathbb{Z}_+^3 : 3x_1 + 3x_2 + x_3 \equiv 2 \pmod 4\}$$

yielding the following matrices of irreducibles:

$$C = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 & 4 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 3 & 4 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Thus, we can write

$$S(2) = \Big\{ x \in \mathbb{Z}_+^3 : x = C\lambda + D\mu,\ \sum_{1 \le i \le 4} \lambda_i = 1,$$
$$\lambda_1, \ldots, \lambda_4 \in \{0, 1\},\ \mu_1, \ldots, \mu_8 \in \mathbb{Z}_+ \Big\},$$

which is the feasible region of the reformulated IP.

By aggregating variables with the same coefficients into a new variable, the reformulation can be written in a more compact way.

In Köppe et al. (2004) two algorithmic schemes that make use of the reformulations are presented. The first one, the dual scheme, computes a group relaxation of constraints that are tight at some fractional point und uses the reformulation to cut off this point. The second one, the primal scheme, starts with an integral tableau and the corresponding basic feasible solution x^0. A reformulation of a group relaxation is then used to update the tableau and to check for a column that provides an augmenting vector v. By pivoting in v in an integer fashion, a new tableau is obtained.

The renewed interest in irreducible solutions for Gomory group relaxations led to the question for further applications in the field of IP theory. In the third section, we present a simultaneous lifting procedure using precomputed tables of irreducible group solutions. First, we describe the general lifting procedure we will refer to in the later section:

Let $P \subset \mathbb{R}_+^{|N|}$ be the convex hull of all feasible solutions to a linear integer programming problem and consider a subset S of the variable index set N of that polytope. Suppose we know that $\sum_{i \in S} \pi_i x_i \le \pi_0$ is a valid inequality of the polytope $P_S = P \cap \{x \in \mathbb{R}^{|N|} : x_j = 0\ \forall j \in N \setminus S\}$. It is a natural question to ask if we can strengthen the inequality by finding coefficients $\pi_j,\ j \in N \setminus S$, such that $\sum_{i \in N} \pi_i x_i \le \pi_0$ is valid for P. The way to determine the missing coefficients is called *lifting*. Especially one is interested in finding facet-defining inequalities that can arise from a lifting procedure. In general, this leads to computing a so-called *maximum* lifting, i.e., a vector $\pi_{N\setminus S}$ such that for no vector $\bar{\pi}_{N\setminus S} \geq \pi_{N\setminus S}$ and distinct from $\pi_{N\setminus S}$ the lifted inequality is valid for P.

There is a polyhedral description of the maximum lifting vectors, but for our purpose we only need the following

Property 3. *Let* $P = \operatorname{conv}\{\, x \in \mathbb{Z}_+^{|N|} : Ax \leq b \,\}$ *be a bounded polyhedron and let* $\sum_{i\in S\subset N} \pi_i x_i \leq \pi_0$ *be a valid inequality for* $P_S = P \cap \{\, x \in \mathbb{R}^{|N|} : x_j = 0 \ \forall j \in N \setminus S \,\}$. *Then* $\pi_{N\setminus S}$ *is a lifting vector if it is contained in the polyhedron*

$$Q = \Big\{ y \in \mathbb{R}^{|N\setminus S|} : \sum_{i\in S} \pi_i x_i + \sum_{j\in N\setminus S} y_j x_j \leq \pi_0 \ \forall x \in P \cap \mathbb{Z}_+^{|N|} \Big\}. \tag{2}$$

We observe that Property 3 just restates the condition that the lifted inequality is valid for the integral points in P.

As the polyhedral description of the lifting vectors requires an enumeration of the integral points in the polytope P, it gives us no hint for a reasonable algorithmic tool to solve the optimization problem. If we could instead restrict ourselves to a (comparatively) small number of vectors necessary to describe the lifting polyhedron this might turn out to be a computationally tractable approach for solving the IP, e.g., by a cutting plane algorithm. We will outline in section 3 a method to accomplish this task. The described method relies on our ability to determine a database for irreducible group solutions.

2 Irreducible group solutions

In this section, we present an algorithm that enables us to compute the irreducible group solutions for cyclic groups up to size 52. The algorithm builds upon some theoretical results valid for irreducible solutions. Throughout this section, we consider the *master cyclic group problem*

$$x_1 + 2x_2 + \cdots + (n-1)x_{n-1} \equiv k \pmod{n} \tag{3}$$

in the additive group of the ring $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. If all irreducible solutions to equation (3) are given, we can extract the solutions of a subsystem of (3) by considering the proper subspace of the variables $x_1, \dots, x_{n-1}$. On the other hand, if we have a group equation where several variables belong to the same group element, we aggregate all these variables into one new integer solution and then reexpand the aggregated solution to the original space of variables, cf. Köppe et al. (2004).

The notion of irreduciblity is compatible with the action of group automorphisms on $\mathbb{Z}_n$. More precisely, if x is an irreducible solution to (3) and ϕ is a group automorphism, then

$$\phi(x) := \big(x_{\phi^{-1}(1)}, \dots, x_{\phi^{-1}(n-1)}\big)$$

is an irreducible solution to the group equation with right hand side $\phi(k)$. Using this property, one can quickly recover the orbit $\text{orb}_n(x) = \{\, \phi(x) : \phi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ is an automorphism $\}$ from a solution x. The main advantage is that one needs to store one solution from an orbit only.

For $\mathbb{Z}_n$, each automorphism of this group is induced by a unit element of this ring, which means it has the form $i \mapsto l \cdot i$ with l being a unit element, i.e., $\gcd(l, n) = 1$. We now prove some statements about the $\|\ \|_1$-norm of irreducible group solutions and about the relationship between homogeneous and inhomogeneous solutions.

Theorem 4. *(i) Let x be a homogeneous irreducible solution to the master cyclic group problem* (3) *in $\mathbb{Z}_n$. Then we have $\|x\|_1 = \sum_{i=1}^{n-1} x_i \le n$.*
(ii) If d is a proper divisor of n and if x is an inhomogeneous irreducible solution with right hand side d, then $\|x\|_1 \le n - d$.
Both types of inequalities are tight.

Proof. (i) We consider an arbitrary decreasing sequence of the form $x = x^0 \ge x^1 \ge \cdots \ge x^k = 0$ where $x^i - x^{i+1} = e^j$ for some $j = 1, \dots, n-1$. As x is irreducible, none of the vectors $x^1, \dots, x^k$ is a homogeneous solution. If two of these vectors, say x^l and x^m, were inhomogeneous solutions with the same right hand side, then the difference $x^l - x^m$ would be a homogeneous solution being less than x, which is impossible. Since there are $n-1$ inhomogeneous residue classes, we have $k \le n-1$, i.e., $\|x\|_1 \le n$.

(ii) We again consider a sequence as we just did. Because of the irreducibility of x, none of the vectors $x^1, \dots, x^k$ is a homogeneous solution nor an inhomogeneous solution with right hand side d. The multiples $2d, \dots, n-d$ of d can occur at most once as the right hand sides of the group equation. Moreover, for each $i = 1, \dots, d-1$ not all of the values $i, i+d, i+2d, \dots, i+n-2d, i+n-d$ can be attained as right hand sides, for otherwise there would be a nonnegative difference of two vectors in the sequence yielding a group solution with right hand side d contradicting the choice of x. Therefore we have

$$k \le \left(\frac{n}{d} - 2\right) + (d-1)\left(\frac{n}{d} - 1\right) = d\left(\frac{n}{d} - 1\right) - 1 = n - d - 1$$

and, consequently, $\|x\|_1 \le n - d$. Of course, the inequality is also valid for the automorphic images of d.

In the first case, the inequality is tight for $x = ne^1$, in the second case $x = (n-d)e^{n-1}$ gives a vector with maximum norm. □

Theorem 5. *Let x be a homogeneous solution with $x_i \ge 1$. Then the following statements are equivalent:*

(i) x is irreducible,
(ii) $x' = x - e^i$ is an irreducible inhomogeneous solution.

In this situation, we call x the inhomogenization *of x', whereas we call x' the* homogenization *of x.*

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. For the converse let us assume that x is reducible, i.e., $x = y+z$ with $y,\ z$ being homogeneous. If we had $y_i > x'_i$ and $z_i > x'_i$ then we would obtain $y_i + z_i \geq x'_i + 2 > x_i$, which is impossible. But otherwise we would have $y \leq x'$ or $z \leq x'$ contradicting the irreducibility of x'. □

The previous theorem shows that the homogenization (or inhomogenization) of a vector preserves irreducibility. If we add an arbitrary unit vector to an inhomogeneous irreducible solution this is no longer true. Instead, the following weaker statement can be made.

Theorem 6. *Let x be an inhomogeneous irreducible solution and assume that $x' = x + e^i$ is reducible for some $i = 1, \ldots, n-1$. If $x' = x^1 + x^h$ is a decomposition of x' with x^h being homogeneous we have:*

(i) $x^1 \leq x$,
(ii) x^1 and x^h are irreducible.

Proof. Part (i). Let us assume that $x^1_i > 0$. Then we had $x^h_i \leq x_i$ and therefore, $x^h \leq x$, which is absurd. It follows that $x^1 \leq x$.

Part (ii). From (i) we immediately obtain the irreducibility of x^1. If x^h was reducible, i.e., $x^h = y + z$, we would again have $y \leq x$ or $z \leq x$. Consequently, x^h is irreducible. □

We now focus on the question of how irreducible group solutions can be (efficiently) computed. Theorem 5 ensures that it is easy to compute the inhomogeneous solutions from the homogeneous ones by simply decreasing positive components by one unit. We can summarize this property in the following

Algorithm 7.

```
01:  Input: set H_n of all irreducible homogeneous solutions of order n;
     Output: set I_n of all irreducible inhomogeneous solutions of order n;
02:  I_n ← ∅;
03:  for all x ∈ H_n do
04:     for all x_i > 0 do
05:        I_n ← I_n ∪ {x − e^i}
06:     end for;
07:  end for;
08:  Output I_n.
```

Theorem 8. *Algorithm 7 is correct.*

Proof. This is an immediate consequence of Theorem 5, because every inhomogeneous vector has a unique homogenization. □

Our computational experiments reveal that Alg. 7 runs fast in practice. Having this in mind, the question of computing irreducible group solutions reduces to computing efficiently the homogeneous vectors. We will present an enumerative algorithm which exploits the group structure of the given problem. The key idea is that we start with the irreducible vectors with $\|\ \|_1$-norm equal to 2, which are precisely the vectors $e^i + e^{n-i}$. By successively adding vectors of the form $e^j + e^k - e^l,\ l \equiv j + k \pmod n$, we compute the vectors up to the norm n according to Theorem 4. We can accelerate this procedure by augmenting only the representatives of the orbits, but using the complete orbits as certificates for reducibility. Let $\mathrm{orb}_n(x)$ denote the orbit of the vector x under the action of the automorphism group of $\mathbb{Z}_n$. In pseudocode, the algorithm reads as follows:

Algorithm 9.

01: Input: n;
Output: set H_n of all irreducible homogeneous solutions of order n and a set R_n of representatives;
02: $H_n \leftarrow \emptyset$;
03: $R_n \leftarrow \emptyset$;
Initialization:
04: **for** $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$ **do**
05:   **if** $x = e^i + e^{n-i} \notin H_n$ **then**
06:     $H_n \leftarrow H_n \cup \mathrm{orb}_n(x)$;
07:     $R_n \leftarrow R_n \cup \{x\}$;
08:   **end if**
09: **end for**;
Main iteration:
10: **for** $i = 2, \ldots, n-1$ **do**
11:   **for all** $x \in R_n \cap \{v : \|v\|_1 = i\}$ **do**
12:     **for all** $y = x + e^j + e^k - e^l \geq \mathbf{0},\ l \equiv j + k \pmod n$, **do**
13:       **if** $\nexists z \in H_n,\ z \leq y$, **then**
14:         $H_n \leftarrow H_n \cup \mathrm{orb}_n(y)$;
15:         $R_n \leftarrow R_n \cup \{y\}$;
16:       **end if**;
17:     **end for**;
18:   **end for**;
19: **end for**;
20: Output $H_n,\ R_n$.

Theorem 10. *Algorithm 9 is correct.*

Proof. Because the set $\{\, v \in \mathbb{Z}_+^{n-1} : \|v\|_1 \leq n \,\}$ is finite, the algorithm terminates. We will show that the algorithm correctly computes the irreducibles for every norm from 2 to n. In the initialization step the algorithm clearly computes all irreducible solutions of norm equal to 2. Let y be an arbitrary irreducible homogeneous solution of norm $\|y\|_1 \geq 3$ and $y \geq e^j + e^k$, and let us assume that all irreducible vectors with a smaller norm have already been computed. Then $x = y - e^j - e^k + e^{j+k \pmod n}$ is an irreducible vector with $\|x\|_1 < \|y\|_1$. Therefore, an automorphic image x' of x must have been added to R_n in line 7 or 15. Considering x' in line 11, the algorithm computes an automorphic image y' of y, so $y \in \mathrm{orb}_n(y') \subset H_n$. For every reducible vector occurring during computation a reducing vector will be found in line 13. Therefore H_n and R_n have the assumed properties. $\square$

Table 1 shows the cardinalities of R_n up to $n = 52$ and the computation times in seconds on a Sun Fire 480R with 1.05 GHz. The database of all irreducible group solutions is available on the Internet (Jach et al., 2004). The total size of the database is 380 MiB (compressed).

Table 1. Cardinalities and computation times for sets of irreducible group solutions

n	$\|R_n\|$	Time	n	$\|R_n\|$	Time	n	$\|R_n\|$	Time
2	1		19	442		36	60520	74
3	2		20	1093		37	34175	47
4	4		21	1109		38	73657	109
5	4		22	1751		39	71537	117
6	11		23	1326		40	119537	217
7	9		24	3769	1	41	76129	151
8	22		25	2489	1	42	234377	541
9	23		26	4951	2	43	111781	280
10	43		27	4682	2	44	238239	662
11	36		28	8372	3	45	246641	745
12	107		29	5893	3	45	344600	1182
13	70		30	19218	12	47	233953	863
14	166		31	9347	7	48	621259	2640
15	200		32	19072	15	49	362268	1652
16	302		33	20441	19	50	791468	3924
17	245		34	31762	31	51	655441	3556
18	693		35	28186	31	52	984646	5653

Remark. In the next section we also consider solutions which have a component for the zero element 0 of $\mathbb{Z}_n$. In this case, we add $1 \cdot 0 \equiv 0 \pmod{n}$ to the set of homogeneous irreducible solutions.

3 Simultaneous lifting using group irreducibles

This section is devoted to demonstrating a novel application of irreducible group solutions to polyhedral combinatorics. We explain how to obtain lifting coefficients from a precomputed table of irreducible homogeneous group solutions. We consider integer programs of the form

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & 0 \le x \le u \\ & x \in \mathbb{Z}^n, \end{aligned} \tag{4}$$

where $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$. In our technique, we will relax the equations $Ax = b$ by reading them modulo a vector $d \in \mathbb{Z}^m$, so that we obtain an equation in the abelian group $G_d = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_m}$. In this process, several columns a^i of the constraint matrix A could be mapped to the same group element; to handle this, we consider a mapping that aggregates the corresponding components x_i. More precisely, we want to consider difference vectors $\bar{x} = x^2 - x^1 \in \mathbb{Z}^n$ of feasible points of (4); therefore, we need to take the sign of the individual components of x into consideration.

Definition 11. Let $A = (a^1, \ldots, a^n) \in \mathbb{Z}^{m\times n}$ be a matrix and let $d \in \mathbb{Z}^m_{>0}$ be a positive integer vector. Let D denote the diagonal matrix where $d = \operatorname{diag}(D)$ such that we have the abelian group $G_d = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_m} \cong \mathbb{Z}^m/L(D)$. We define a mapping $f_d \colon \mathbb{Z}^n \to \mathbb{Z}_+^{|G_d|}$ in the following way: Let $\bar{x} \in \mathbb{Z}^n$. In order to define $f_d(\bar{x}) = \hat{x}$ we identify each column vector $\operatorname{sgn}(\bar{x}_i)a^i$ (where $\operatorname{sgn}(0) = 0$) with its canonical image in the group G_d and aggregate the components $\bar{x}_i$ of $\bar{x}$ whose matrix columns have the same image g in the component $\hat{x}_g$, i.e., we have

$$\hat{x}_g = \sum_{i\,:\,\operatorname{sgn}(\bar{x}_i)a^i \,\equiv\, g \pmod{D}} |\bar{x}_i| \quad \forall g \in G_d.$$

We will call the vector $\hat{x}$ the *group image* of the solution $\bar{x}$ with respect to the modulus vector d.

Obviously, the above defined group image yields a homogeneous group solution in the group G_d (with an additional component for the zero element in G_d) if $A\bar{x} = (\lambda_1 d_1, \ldots, \lambda_m d_m)$ for some $\lambda \in \mathbb{Z}^m$. Conversely, every homogeneous group solution $\hat{x}$ can be mapped back to some vector $\bar{x}$ satisfying $A\bar{x} = (\lambda_1 d_1, \ldots, \lambda_m d_m)$, provided that each positive component of $\hat{x}$ corresponds to some column $\pm a^i$ of the matrix A and no column is used with both signs.

In the next theorem we show that it suffices to consider preimages of irreducible group solutions in order to obtain a polyhedral description for the lifting vectors.

Theorem 12. *Consider the polytope*

$$P = \operatorname{conv}\{\, x \in \mathbb{Z}^n : Ax = b,\ 0 \le x \le u \,\}$$

with integral data A, b, u. For a subset S of the index set N let x^* be a point in $P_S = P \cap \{x \in \mathbb{Z}^n_+ : x_j = 0\ \forall j \in N \setminus S\}$. Further let $\sum_{i\in S} \pi_i x_i \le \pi_0$ be a valid inequality for P_S which is tight for x^*. Then, given any $d \in \mathbb{Z}^m_{>0}$, every feasible point in the polyhedron

$$\tilde{Q}_d = \{\, y \in \mathbb{R}^{|N\setminus S|} : y^\top z_{N\setminus S} + \pi_S^\top z_S \le 0\ \forall z \in \mathbb{Z}^n, 0 \le x^* + z \le u, \text{such that } \hat{z} = f_d(z) \text{ is an irreducible solution to } \textstyle\sum_{g\in G_d} \hat{z}_g g = 0 \,\} \tag{5}$$

is a lifting vector.

Proof. We will show $\tilde{Q}_d \subset Q$ where Q denotes the polyhedron defined in equation (2). To this end, let $y \in \tilde{Q}_d$. We show that

$$\sum_{i\in S} \pi_i x_i + \sum_{j\in N\setminus S} y_j x_j \leq \pi_0 \quad \text{for } x \in P \cap \mathbb{Z}^n.$$

Let x be an arbitrary vector in $P \cap \mathbb{Z}^n$ and let D be the diagonal matrix where $d = \operatorname{diag}(D)$. For the difference vector $\bar{x} = x - x^*$ we have $A\bar{x} = 0$. Thus, the group image $f_d(\bar{x})$ is a homogeneous group solution which can be decomposed into a sum of irreducible solutions, say $f_d(\bar{x}) = \sum_{j=1}^k h^j$. Since we have

$$(f_d(\bar{x}))_g = \sum_{i\,:\,\operatorname{sgn}(\bar{x}_i)a^i \,\equiv\, g \pmod{D}} |\bar{x}_i| = \sum_{j=1}^k h_g^j \quad \forall g \in G_d,$$

we can partition each number $|\bar{x}_i|$, where $\operatorname{sgn}(\bar{x}_i)a^i \equiv g \pmod{D}$, into a sum $|\bar{x}_i| = \sum_{j=1}^k |\bar{x}_i^j|$ such that each $\bar{x}_i^j$ has the same sign as $\bar{x}_i$ or is equal to 0, and that

$$\sum_{i\,:\,\operatorname{sgn}(\bar{x}_i)a^i \,\equiv\, g \pmod{D}} |\bar{x}_i^j| = h_g^j \quad \forall j = 1,\dots,k,\ \forall g \in G_d.$$

By the definition of the numbers $\bar{x}_i^j$, we see that for each h^j we have a preimage $z^j = (\bar{x}_1^j,\dots,\bar{x}_n^j)$ such that $\sum_{j=1}^k z^j = \bar{x}$ and that $\bar{x},\ z^1,\dots,\ z^k$ all lie in the same orthant of $\mathbb{R}^n$.

From $0 \leq x^* + \bar{x} \leq u$ and the orthant condition we immediately obtain $0 \leq x^* + z^j \leq u$ for each z^j, which therefore yields a constraint in the definition of $\tilde{Q}_d$. Because $y \in \tilde{Q}_d$, we have

$$y^\top z_{N\setminus S}^j + \pi_S^\top z_S^j \leq 0 \quad \text{for } j = 1,\dots,k. \tag{6}$$

Adding up the inequalities (6) and the equation $\pi_S^\top x_S^* = \pi_0$ gives

$$y^\top x_{N\setminus S} + \pi_S^\top x_S = y^\top \left(\sum_{j=1}^k z_{N\setminus S}^j\right) + \pi_S^\top \left(\sum_{j=1}^k z_S^j + x_S^*\right) \leq \pi_0,$$

completing the proof. □

Remark 13. Because the number of irreducible group solutions seems to grow exponentially with the group order, the polyhedral description (5) is of exponential size. Clearly lifting coefficients could also be obtained from the linear program proposed by Gomory (1969), which only has $\Theta(|G_d|^2)$ constraints. However, the lifting procedure of Theorem 12 considers a discrete relaxation of the integer program that consists of a group relaxation and bound constraints. The lifting coefficients are optimal for this discrete relaxation. This means that the liftings obtained from it are potentially stronger than those that could be obtained from a pure group relaxation (for the same group order), especially for problems with very low variable bounds.

We next illustrate the lifting procedure on two knapsack problem instances.

Example 14. Let us consider the problem

$$\begin{aligned} P &= \operatorname{conv}\{\, x \in \{0,1\}^5 : x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 9 \,\} \\ &= \operatorname{conv}\{(0,0,0,1,1),\ (0,1,1,1,0),\ (1,0,1,0,1)\}. \end{aligned}$$

We can take $x^* = (0,0,0,1,1)^\top$ and the inequality $x_4 + x_5 \le 2$ valid for $P_S = P \cap \{\, x \in \mathbb{R}^5 : x_1 = x_2 = x_3 = 0 \,\}$. Without having chosen a modulus yet, we already know that we only have to consider vectors z in the orthant $\{x_1,\ x_2,\ x_3 \ge 0,\ x_4,\ x_5 \le 0\}$ because starting from x^* we cannot decrease x_1, x_2, and x_3, nor increase x_4 and x_5.

If we take the modulus $m = 3$, for instance, the resulting system of inequalities consists of the inequalities $y_3 \le 0$, $y_1 + y_2 \le 0$, $y_1 \le 1$, $y_2 \le 1$, and $-2 \le 0$ coming from the vectors $z^1 = (0,0,1,0,0)$, $z^2 = (1,1,0,0,0)$, $z^3 = (1,0,0,-1,0)$, $z^4 = (0,1,0,0,-1)$, and $z^5 = (0,0,0,-1,-1)$, respectively. Hence, we get the lifting coefficients $(1,-1,0)$ and $(-1,1,0)$ (or any convex combination of these two). The resulting inequalities $x_1 - x_2 + x_4 + x_5 \le 2$ and $-x_1 + x_2 + x_4 + x_5 \le 2$ are valid for P, but they do not define facets of P.

If we select $m = 7$, the only possibilities for choosing a vector z are $z^1 = (1,0,1,-1,0)$ and $z^2 = (0,1,1,0,-1)$ yielding the system $y_1 + y_3 \le 1$ and $y_2 + y_3 \le 1$. As a result for the lifting coefficients we obtain the solutions $y^1 = (1,1,0)$ and $y^2 = (0,0,1)$ (or any convex combination of these two). The inequalities $x_1 + x_2 + x_4 + x_5 \le 2$ and $x_3 + x_4 + x_5 \le 2$ both are facets of P.

Example 15. Let $P = \operatorname{conv}\{x \in \mathbb{Z}_+^4 : 3x_1 + 5x_2 + 17x_3 + 23x_4 = 94,\ x \le (3,5,5,4)^\top\}$. For $x^* = (3,0,5,0)$ we have the inequality $x_1 + x_3 \le 8$ valid for $P_S = P \cap \{x \in \mathbb{R}^5 : x_2 = x_4 = 0\}$. If we take $m = 29$ as the modulus, the system of inequalities in $\tilde{Q}_m$ reduces to

$$\begin{aligned} y_4 &\le \tfrac{1}{2}, \\ y_2 \qquad &\le \tfrac{1}{2}, \\ 3y_2 + 2y_4 &\le 1. \end{aligned}$$

Nonnegative feasible solutions are $y^1 = (\frac{1}{3}, 0)$ and $y^2 = (0, \frac{1}{2})$ yielding the following lifted inequalities:

$$\begin{aligned} 3x_1 + x_2 + 3x_3 \qquad &\le 24, \\ 2x_1 + \qquad 2x_3 + x_4 &\le 16. \end{aligned}$$

Both inequalities do not define a facet of P, but the first one cuts off the fractional point $(3, 5, \frac{60}{17}, 0)$ of the LP relaxation of P. The second inequality does not cut off a fractional point.

In general, the lifting procedure via group solutions can be used to obtain nontrivial lifting coefficients. In particular, this applies to the case when it is not known if the original inequality, which is valid for the subspace polytope only, is also valid for the polytope P. Below we experiment with instances of the kind

$$P = \operatorname{conv}\{\, x \in \mathbb{Z}_+^n : a^\top x = b,\ x \le u \,\} \tag{7}$$

with $a \in \mathbb{Z}^n,\ u \in \mathbb{Z}_+^n$, and $b \in \mathbb{Z}$.

For each such instance we generate a facet of an appropriate subspace polytope P_S and then apply the lifting algorithm based on our group database. More precisely, we follow

Algorithm 16.
Input: An IP polytope P, a valid inequality for a subspace polytope P_S which is tight at a point $x^* \in P_S$.
Output: A lifted inequality valid for P, or *failure*.

1. Fix a modulus m for the group relaxation.
2. For the modulus m read off the irreducible group solutions possible as group images from the database.
3. Generate the polyhedron $\tilde{Q}_d = \tilde{Q}_m$ as defined in equation (5).
4. Select an auxiliary objective function $w \in \mathbb{R}^{|N \setminus S}$ and solve the linear program $\max\{\, w^\top y : y \in \tilde{Q}_m \,\}$ to determine a vertex y of $\tilde{Q}_m$. Return the optimal solution y if it exists; otherwise return *failure*.

We apply the algorithm to a variety of examples where we experiment with the following parameters: the dimension $n \in \{10, 15, 20, 30\}$ of the problem and the uniform upper bound $u \in \{1, 2\}$ on the variables. In order to simplify our computational efforts, we always choose the subspace polytope to lie in the space spanned by the first three variables, and the right hand side b is chosen in a way that $(x_1, x_2, x_3) = (1, 1, 1)$ is a feasible solution for which the original inequality is tight.

In the case $u = 1$ we consider both an original inequality that is valid for P,

$$x_1 + x_2 + x_3 \le 3, \tag{8}$$

and one which is not valid for P,

$$x_1 - x_2 \le 0, \tag{9}$$

whereas in the case $u = 2$ we only consider inequality (8), which is not valid for P in this case.

For each choice of parameters, we randomly generate 30 feasible instances of the 1-row equality problem (7). The coefficients $a_1, \dots, a_n$ are randomly chosen between -50 and 50. Then the right-hand side is chosen in a way that the point $(1, 1, 1)$ is a feasible solution. For all generated instances, we checked that the inequality (9) is indeed not valid.

In our experiment, we wish to investigate how the quality of the lifting procedure depends on the choice of the modulus m. The larger the modulus, the more irreducible group solutions exist, so the linear programs to be solved for obtaining the lifting coefficients become larger, thus harder to solve. In order to see whether the increased computational effort pays off, we carry out the computations for all moduli m in the range from 2 to 30. For a given modulus m, the polyhedron $\tilde{Q}_m$ can be empty or non-empty. The former case occurs when in the definition (5) of the polyhedron $\tilde{Q}_d$ there occurs a vector z that has non-zero components in the set S only; such a vector can lead to a contradictory constraint $\pi_S^\top z_S \leq 0$.

In the case of a non-empty polyhedron $\tilde{Q}_m$, every vertex of $\tilde{Q}_m$ yields a valid lifting vector. We evaluate the strength of the corresponding lifted inequality by computing whether the inequality cuts off a fractional point of the LP relaxation of P. If this happens to be true, we regard the lifted inequality as nontrivial.

Example 17. We first show one of the generated instances for dimension $n = 10$. We consider the polyhedron

$$P = \operatorname{conv}\{\, x \in \mathbb{Z}_+^{10} : 9x_1 + 14x_2 + 16x_3 + 39x_4 + 2x_5 + 28x_6 + 11x_7 \\ - 20x_8 - 13x_9 - 37x_{10} = 39,\ x_i \leq u\ \forall i = 1, \dots, 10 \,\}$$

where $u = 1$. For the feasible point $(x_1, x_2, x_3) = (1, 1, 1)$ of the subspace polyhedron $P_{\{1,2,3\}}$, we start with the inequality $x_1 + x_2 + x_3 \leq 3$. In Table 2 we show the lifting coefficients that we obtain for the moduli $m = 3$, 6, and 17, using four randomly generated auxiliary objective functions,

$$\begin{aligned} c^1 &= (17, 34, 10, 49, 25, 43, 39, 33, 37, 16)^\top, \\ c^2 &= (28, 45, 11, 16, 17, \ \ 4, 31, 45, 32, 36)^\top, \\ c^3 &= (\ \ 5, 29, 44, 10, 25, 43, 14, 39, 38, 21)^\top, \\ c^4 &= (37, 29, 40, 11, 46, 33, \ \ 8, 13, 31, \ \ 1)^\top. \end{aligned}$$

The results in Table 2 show that we can obtain both trivial and non-trivial lifting coefficients, depending on the choice of the objective function. A higher modulus seems to lead to a higher proportion of non-trivial lifting coefficients.

To verify this observation, we consider the optimal vertices for a random sampling of auxiliary objective functions in order to measure which proportion of the possible vertices yields nontrivial lifting coefficients. To this end, we randomly generate 40 objective functions by choosing the objective coefficients uniformly from the set $\{0, 1, \dots, 50\}$. (We remark that if Algorithm 16 were to be applied as a separation procedure, the auxiliary objective function should be chosen in a way depending on the point to be cut off, rather than randomly.)

The computational results of our experiment are shown in Tables 3–6. For the LP calculations we use CPLEX, version 6.6.

In all tables, the column “# cuts” shows the number of instances (out of 30) where at least for one auxiliary objective function we obtain a lifting vector yielding a cut-off. In the column “# inf.”, the number of instances where $\tilde{Q}_m$ is empty

Table 2. Lifting coefficients in Example 17 for a few auxiliary objective functions and moduli

Auxiliary objective	Cut coefficients										
	π_1	π_2	π_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	
					$m = 3$						
c^1	1	1	1	0	0	0	0	0	0	0	trivial
c^2	1	1	1	0	0	0	0	0	0	0	trivial
c^3	1	1	1	0	0	0	0	0	0	0	trivial
c^4	1	1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	-1	cut-off
					$m = 6$						
c^1	1	1	1	0	0	0	0	0	0	0	trivial
c^2	1	1	1	$-\frac{3}{4}$	$-\frac{1}{2}$	-1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	cut-off
c^3	1	1	1	0	0	0	0	0	0	0	trivial
c^4	1	1	1	-1	$\frac{1}{2}$	$-\frac{1}{2}$	-2	$-\frac{1}{2}$	$\frac{3}{2}$	-2	cut-off
					$m = 17$						
c^1	1	1	1	0	$-\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	cut-off
c^2	1	1	1	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$	$\frac{2}{3}$	$-\frac{5}{3}$	$\frac{2}{3}$	cut-off
c^3	1	1	1	$-\frac{5}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{1}{3}$	$-\frac{2}{3}$	cut-off
c^4	1	1	1	3	4	-2	-2	-3	-2	-3	cut-off

(i.e., the auxiliary LP is infeasible) are listed. Finally, the column "% cuts" mirrors the percentage how often a nontrivial cut has been found (relative to the number of nonempty $\tilde{Q}_m$ multiplied by the number of objectives), i.e., whenever a lifting vector can be determined, we measure how often this lifting is nontrivial.

The computational results show that it is possible to obtain nontrivial lifting coefficients using the proposed technique. For original inequalities that are valid for the complete problem (ineq. (8) for $u = 1$), the algorithm only works well for small dimensions $n \leq 15$. For original inequalities that are not valid for the complete problem (ineq. (9) for $u = 1$ and (8) for $u = 2$), Algorithm 16 computes nontrivial lifting coefficients in the majority of the randomly generated instances, if the modulus m is chosen large enough. We note, however, that the quality of the lifting procedure becomes worse for larger upper bounds.

Conclusions. We conclude that the lifting procedure works best for problems with small upper bounds, when we try to lift inequalities for the subspace problem that are not valid for the complete problem. For problems where the variables have larger upper bounds, we believe that the strength of the inequalities obtained with our lifting procedure is essentially the same as the subadditive cuts obtained from the linear program of Gomory (1969). Thus, the overhead of our lifting procedure (which considers auxiliary problems with an exponential number of inequalities, rather than a quadratic number of inequalities) cannot pay off. Moreover, experiments in the past have shown that group relaxations for small group orders (like $m \leq 30$ in our experiments) are very weak. Recent experiments by Fischetti and Saturni (2005) also show that one needs group orders of at least 20 to reach the same strength as Gomory mixed-integer cuts.

We also need to point out that the computational experiments carried out in this paper are very limited. We believe that the computational value of our lifting procedure can only be determined by extensive tests within a branch-and-cut system. Also more interesting problem classes than knapsack problems should be addressed. Such experiments, however, are beyond the intended scope of this paper.

If one tries to use our lifting procedure in a branch-and-cut system, many relevant and interesting questions will arise:

- How should one choose the inequalities to be lifted?
- Which is the choice of the modulus m that gives the best trade-off between the strength of the inequalities and the size of the auxiliary problems and of the required database?
- Instead of writing down and solving an auxiliary LP of exponential size, a separation procedure should be designed.
- In a practical implementation, it may also be essential to strengthen the discrete relaxation even more, for instance by taking generalized upper bound constraints into consideration.

We believe that, if these questions are solved, our lifting procedure could be of practical use in a branch-and-cut system, especially for 0/1 problems.

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Table 3. Lifting results for dimension $n = 10$

	$u = 1$, ineq. (8)			$u = 1$, ineq. (9)			$u = 2$, ineq. (8)		
m	# cuts	# inf.	% cuts	# cuts	# inf.	% cuts	# cuts	# inf.	% cuts
2	30	0	47	7	23	100	0	30	0
3	30	0	55	9	21	100	0	30	0
4	30	0	74	17	13	100	0	30	0
5	30	0	82	14	16	100	1	29	100
6	30	0	90	16	14	95	1	29	100
7	30	0	94	21	9	99	5	25	100
8	30	0	96	19	11	99	7	23	100
9	30	0	97	22	8	97	4	26	100
10	30	0	98	24	6	97	8	22	100
11	30	0	99	28	2	98	13	17	100
12	30	0	99	20	10	99	8	22	100
13	30	0	99	26	4	99	16	14	100
14	30	0	100	25	5	99	17	13	100
15	30	0	100	21	9	100	6	24	100
16	30	0	100	26	4	99	16	14	100
17	30	0	99	26	4	100	12	18	100
18	30	0	99	27	3	99	12	18	100
19	30	0	99	30	0	97	16	14	100
20	30	0	99	29	1	99	18	12	100
21	30	0	99	27	3	98	16	14	100
22	30	0	100	30	0	99	21	9	100
23	30	0	99	28	2	99	21	9	100
24	30	0	99	23	7	99	19	11	100
25	30	0	99	29	1	99	21	9	100
26	30	0	100	29	1	100	24	6	100
27	30	0	99	30	0	99	19	11	100
28	30	0	99	28	2	100	23	7	100
29	30	0	99	28	2	99	25	5	100
30	30	0	100	26	4	99	18	12	100

Table 4. Lifting results for dimension $n = 15$

	$u = 1$, ineq. (8)			$u = 1$, ineq. (9)			$u = 2$, ineq. (8)		
m	# cuts	# inf.	% cuts	# cuts	# inf.	% cuts	# cuts	# inf.	% cuts
2	16	0	10	7	23	100	0	30	0
3	17	0	9	15	15	100	0	30	0
4	24	0	15	17	13	100	0	30	0
5	24	0	10	17	13	100	2	28	100
6	26	0	23	19	11	100	3	27	100
7	26	0	19	25	5	100	7	23	100
8	27	0	26	21	9	100	6	24	100
9	29	0	29	24	6	100	9	21	100
10	30	0	36	24	6	100	9	21	100
11	30	0	37	27	3	100	12	18	100
12	30	0	42	26	4	100	10	20	100
13	30	0	36	26	4	100	9	21	100
14	30	0	40	26	4	100	19	11	100
15	30	0	45	21	9	100	10	20	100
16	30	0	44	27	3	100	12	18	100
17	30	0	44	25	5	100	13	17	100
18	30	0	53	28	2	100	16	14	100
19	30	0	44	30	0	100	21	9	100
20	30	0	50	29	1	100	19	11	100
21	30	0	56	30	0	100	20	10	100
22	30	0	62	29	1	100	19	11	100
23	30	0	63	27	3	100	17	13	100
24	30	0	64	26	4	100	21	9	100
25	30	0	68	28	2	100	20	10	100
26	30	0	64	28	2	99	20	10	100
27	30	0	68	28	2	100	22	8	100
28	30	0	64	28	2	100	23	7	100
29	30	0	67	27	3	100	21	9	100
30	30	0	73	25	5	100	18	12	100

Table 5. Lifting results for dimension $n = 20$

	$u = 1$, ineq. (8)			$u = 1$, ineq. (9)			$u = 2$, ineq. (8)		
m	# cuts	# inf.	% cuts	# cuts	# inf.	% cuts	# cuts	# inf.	% cuts
2	5	0	6	8	22	100	0	30	0
3	3	0	< 1	12	18	100	0	30	0
4	6	0	6	21	9	100	0	30	0
5	1	0	< 1	16	14	100	2	28	100
6	9	0	6	21	9	100	2	28	100
7	5	0	< 1	21	9	100	7	23	100
8	8	0	6	24	6	100	8	22	100
9	9	0	< 1	25	5	100	4	26	100
10	7	0	6	22	8	100	9	21	100
11	9	0	1	30	0	100	12	18	100
12	13	0	6	26	4	100	10	20	100
13	7	0	1	25	5	100	10	20	100
14	10	0	6	25	5	100	17	13	100
15	10	0	3	22	8	100	5	25	100
16	11	0	7	28	2	100	15	15	100
17	7	0	2	26	4	100	11	19	100
18	11	0	7	28	2	100	14	16	100
19	6	0	< 1	28	2	100	19	11	100
20	15	0	7	26	4	100	19	11	100
21	8	0	1	26	4	100	13	17	100
22	16	0	9	30	0	100	22	8	100
23	17	0	3	26	4	100	19	11	100
24	16	0	7	28	2	100	22	8	100
25	16	0	2	27	3	100	20	10	100
26	17	0	8	28	2	100	19	11	100
27	18	0	3	30	0	100	22	8	100
28	18	0	8	28	2	100	26	4	100
29	20	0	3	25	5	100	21	9	100
30	17	0	11	25	5	100	17	13	100

Table 6. Lifting results for dimension $n = 30$

	$u = 1$, ineq. (8)			$u = 1$, ineq. (9)			$u = 2$, ineq. (8)		
m	# cuts	# inf.	% cuts	# cuts	# inf.	% cuts	# cuts	# inf.	% cuts
2	1	0	< 1	5	25	100	0	30	0
3	0	0	0	13	17	100	0	30	0
4	1	0	< 1	15	15	100	0	30	0
5	0	0	0	12	18	100	0	30	0
6	0	0	0	17	13	100	0	30	0
7	0	0	0	21	9	100	7	23	100
8	1	0	< 1	19	11	100	7	23	100
9	0	0	0	23	7	100	4	26	100
10	1	0	< 1	20	10	100	7	23	100
11	0	0	0	28	2	100	10	20	100
12	1	0	< 1	25	5	100	8	22	100
13	0	0	0	28	2	100	10	20	100
14	1	0	< 1	24	6	100	14	16	100
15	0	0	0	21	9	100	9	21	100
16	0	0	0	25	5	100	13	17	100
17	0	0	0	28	2	100	13	17	100
18	1	0	< 1	27	3	100	11	19	100
19	0	0	0	29	1	100	18	12	100
20	1	0	< 1	26	4	100	19	11	100
21	0	0	0	29	1	100	18	11	100
22	1	0	< 1	29	1	100	18	12	100
23	0	0	0	30	0	100	22	8	100
24	1	0	< 1	26	4	100	17	13	100
25	0	0	0	27	3	100	19	11	100
26	0	0	0	30	0	100	20	10	100
27	0	0	0	29	1	100	23	7	100
28	1	0	< 1	27	3	100	23	7	100
29	0	0	0	26	4	100	22	8	100
30	1	0	< 1	25	5	100	16	14	100

References

Julián Aráoz, Lisa Evans, Ralph E. Gomory, and Ellis L. Johnson. Cyclic group and knapsack facets. *Mathematical Programming*, B96:377–408, 2003.

Matteo Fischetti and Cristiano Saturni. Mixed-integer cuts from cyclic groups. In Michael Jünger and Volker Kaibel, editors, *IPCO*, volume 3509 of *Lecture Notes in Computer Science*, pages 1–11. Springer, 2005. ISBN 3-540-26199-0. Available electronically as `http://dx.doi.org/10.1007/11496915_1`.

Ralph E. Gomory. Some polyhedra related to combinatorial problems. *Linear Algebra and Applications*, 2:451 – 558, 1969.

Ralph E. Gomory and Ellis L. Johnson. T-space and cutting planes. *Mathematical Programming*, B96: 341–375, 2003.

Ralph E. Gomory and Ellis L. Johnson. Some continuous functions related to corner polyhedra I. *Mathematical Programming*, 3:23 – 85, 1972a.

Ralph E. Gomory and Ellis L. Johnson. Some continuous functions related to corner polyhedra II. *Mathematical Programming*, 3:359 – 389, 1972b.

Ralph E. Gomory, Ellis L. Johnson, and Lisa Evans. Corner polyhedra and their connection with cutting planes. *Mathematical Programming*, B96:321–339, 2003.

Raymond Hemmecke. On the computation of Graver bases using symmetries. Manuscript, 2004.

Matthias Jach, Matthias Köppe, and Robert Weismantel. Tables of irreducible group solutions, 2004. available from `http://www.math.uni-magdeburg.de/~mkoeppe/art/ppi/group.html`.

Matthias Köppe, Quentin Louveaux, Robert Weismantel, and Laurence A. Wolsey. Extended formulations for Gomory corner polyhedra. *Discrete Optimization*, 1:141–165, 2004.